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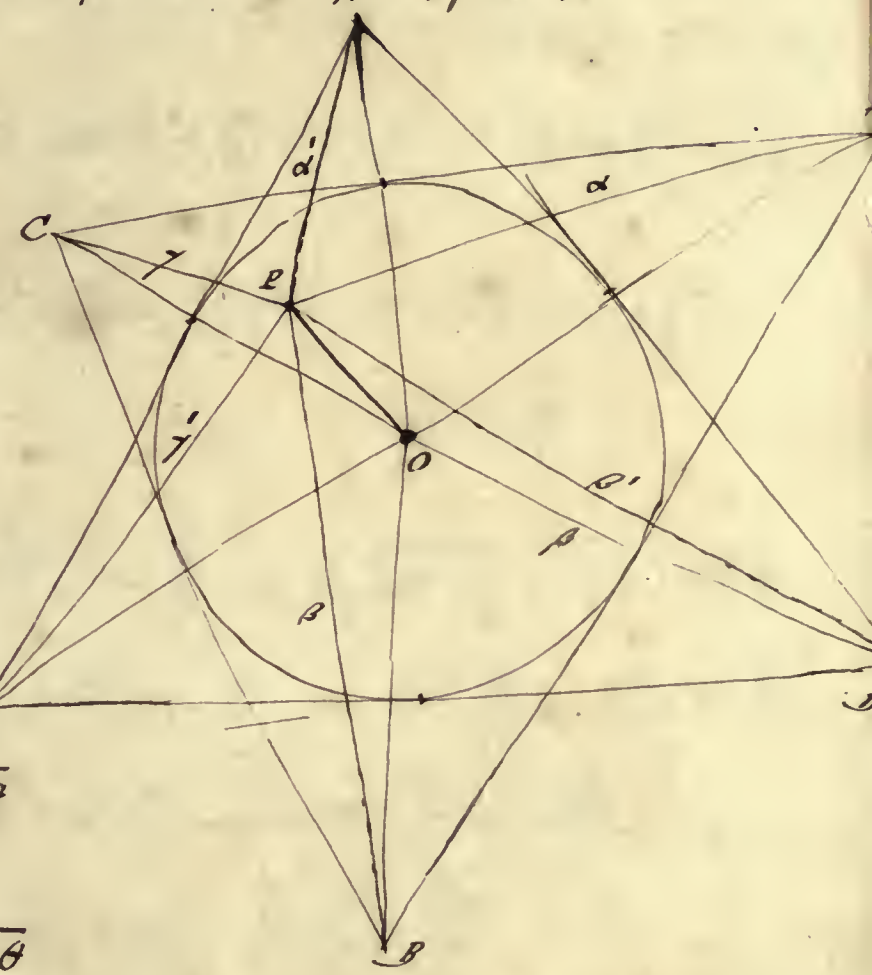
[Mathematical problems
Vol. 2]

# Trigonometrical Problems

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If the centre of the inscribed ⊙ of a triangle be fixed and $\alpha.\beta.\gamma$ represent the distances of its angles from any other fixed point then will $a\alpha^2+b\beta^2+c\gamma^2$ be constant for all positions of the Δ.




$ABC$ the initial posit. of the Δ.
$O$ the cent. of the inscribed ⊙
$P$ any fixed point
$A'B'C'$ the posit. of the Δ after it has revolved thro' any angle $\theta$

$\angle POA' = \phi \therefore \angle POA = \overline{\phi+\theta}$
$\angle POC' = \psi \therefore \angle POC = \overline{\psi-\theta}$
$\angle POB' = \chi \therefore \angle POB = \overline{\chi+\theta}$

Now.

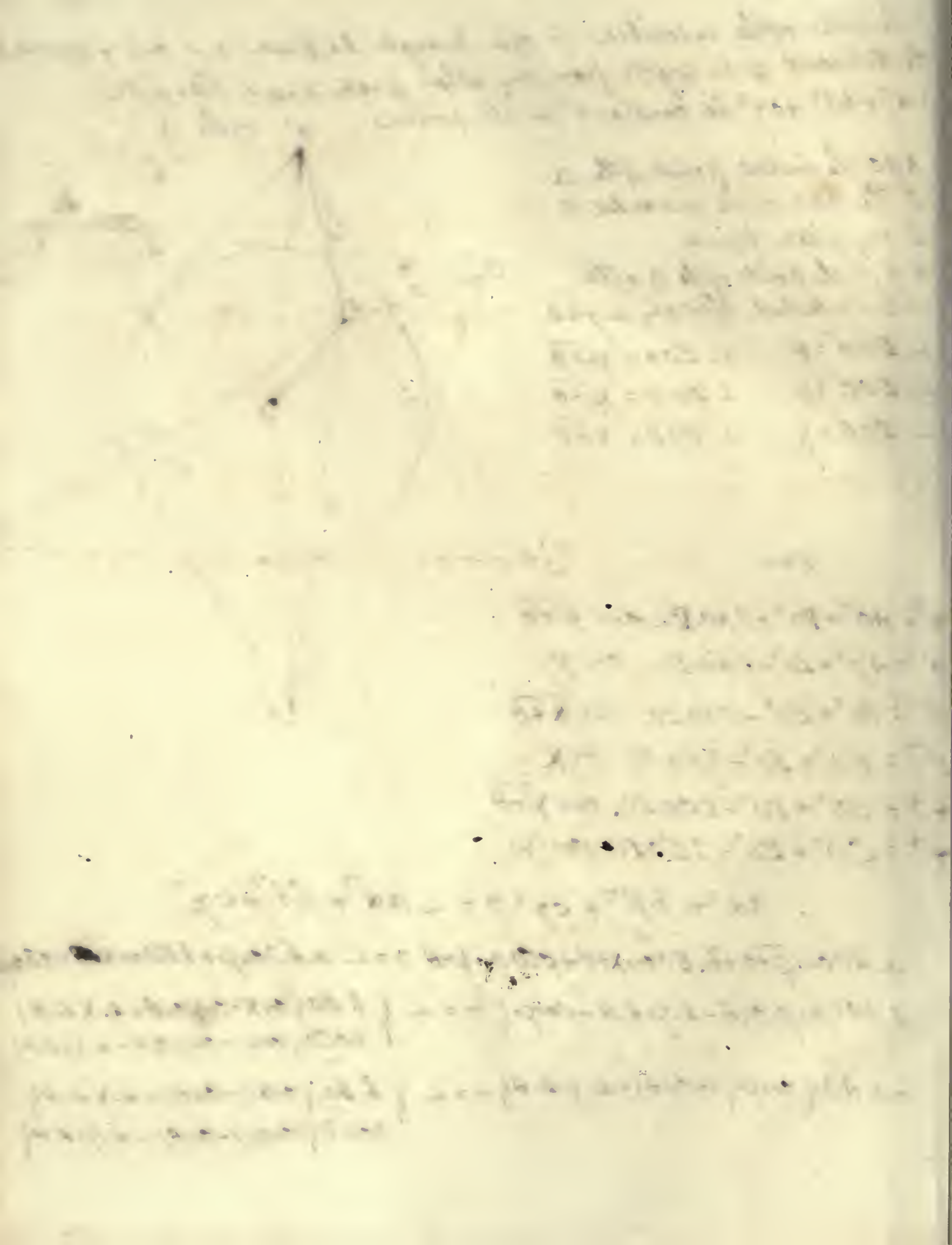
$\alpha^2 = AO^2+PO^2-2AO.PO.\cos\overline{\phi+\theta}$
$\alpha'^2 = A'O^2+PO^2-2A'O.PO.\cos\phi.$
$\beta^2 = BO^2+PO^2-2BO.PO.\cos\overline{\chi+\theta}$
$\beta'^2 = B'O^2+PO^2-2B'O.PO.\cos\chi.$
$\gamma^2 = CO^2+PO^2-2CO.PO.\cos\overline{\psi-\theta}$
$\gamma'^2 = C'O^2+PO^2-2C'O.PO.\cos\psi.$

$\therefore a\alpha^2+b\beta^2+c\gamma^2 = \angle a\alpha'^2+b\beta'^2+c\gamma'^2.$

$aAO\cos\overline{\phi+\theta}+b.BO\cos\overline{\chi+\theta}+cCO\cos\overline{\psi-\theta} = \angle aA'O\cos\phi+bB'O\cos\chi+cC'O\cos\psi$

$aAO\{\cos\phi\cos\theta-\sin\phi\sin\theta-\cos\phi\} = \angle \begin{cases} b.BO\{\cos\chi-\cos\chi\cos\theta+\sin\chi\sin\theta\} \\ +cCO\{\cos\psi-\cos\psi\cos\theta-\sin\psi\sin\theta\} \end{cases}$

$-aAO\{\cos\phi(1-\cos\theta)-\sin\phi\sin\theta\} = \angle \begin{cases} b.BO.\{\cos\chi(1-\cos\theta)+\sin\chi\sin\theta\} \\ +c.CO\{\cos\psi(1-\cos\theta)-\sin\psi\sin\theta\} \end{cases}$



$$-a.AO.\left\{\cos\frac{\theta}{2}\sin\varphi+\sin\frac{\theta}{2}\cos\varphi\right\}>=<\left\{\begin{array}{l}b.BO\left(\cos X\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\sin X\right)\\ +c.CO\left\{\cos\psi\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\sin\psi\right\}\end{array}\right.$$

$$-a\,AO\sin\left(\varphi+\frac{\theta}{2}\right)>=<b.BO.\sin\left(X+\frac{\theta}{2}\right)+c.CO.\sin\left(\psi+\frac{\theta}{2}\right)\quad(1)$$

---

Now $X+\psi=2\pi-C'OB'=2\pi-\left\{\pi-\frac{B+C}{2}\right\}=2\pi-\left\{\pi-\frac{\pi}{2}+\frac{A}{2}\right\}$

$$=\frac{3\pi}{2}-\frac{A}{2}.$$

$$\therefore\ \psi=\frac{3\pi}{2}-\frac{A}{2}-X.$$

$$\therefore\ \sin\left(\psi-\frac{\theta}{2}\right)=\sin\left(\frac{3\pi}{2}-\frac{A}{2}-X-\frac{\theta}{2}\right)=\sin\left\{2\pi-\frac{A}{2}-\frac{\pi}{2}+X+\frac{\theta}{2}\right\}$$

$$=\sin 2\pi-\frac{A}{2}\cos\frac{\pi}{2}+X+\frac{\theta}{2}-\cos 2\pi-\frac{A}{2}\sin\frac{\pi}{2}+X+\frac{\theta}{2}$$

$$=\cos X+\frac{\theta}{2}\cos\frac{A}{2}-\sin X+\frac{\theta}{2}\sin\frac{A}{2}.\quad(2).$$

Also. $X-\varphi=\frac{\pi}{2}+\frac{C}{2}\quad\therefore\ \varphi=X-\frac{\pi}{2}-\frac{C}{2}$

$$\therefore\ \sin\varphi+\frac{\theta}{2}=\sin\left\{X+\frac{\theta}{2}-\frac{\pi}{2}+\frac{C}{2}\right\}$$

$$=\sin X+\frac{\theta}{2}\cos\frac{\pi}{2}+\frac{C}{2}-\cos X+\frac{\theta}{2}\sin\frac{\pi}{2}+\frac{C}{2}$$

$$=\sin X+\frac{\theta}{2}\sin\frac{C}{2}-\cos X+\frac{\theta}{2}\cos\frac{C}{2}.\quad(3).$$

Substituting in (1) the values of $\sin\varphi+\frac{\theta}{2}$. $\sin\psi-\frac{\theta}{2}$ found in (2) and (3) we have.

$$a\alpha^2 + b\beta^2 + c\gamma^2 \gamma = < a\alpha'^2 + b\beta'^2 + c\gamma'^2$$

$$a.AO.\left\{\begin{array}{l}\cos\overline{X+\frac{\theta}{2}}\cos\frac{C}{2}\\ +\sin\overline{X+\frac{\theta}{2}}\,a.\frac{c}{2}\end{array}\right\}\gamma = < \left\{\begin{array}{l} b.BO\sin\overline{X+\frac{\theta}{2}}\\ +cCO\left\{\cos\overline{X+\frac{\theta}{2}}\cos\frac{A}{2} - \sin\overline{X+\frac{\theta}{2}}\sin\frac{A}{2}\right\}\end{array}\right.$$

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by similar value $AO = r\,\mathrm{cosec}\,\frac{A}{2} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}\cdot\sqrt{\frac{bc}{(s-b)(s-c)}}$

$$= \sqrt{\frac{(s-a).bc}{s}}$$

$$\text{also } \cos\frac{C}{2} = \sqrt{\frac{s.(s-c)}{(s-a)(s-b)}} = \sqrt{\frac{s.(s-c)}{ab}},\quad \sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$\therefore aAO\cos\frac{C}{2} = a\sqrt{\frac{(s-a).bc}{s}}\cdot\sqrt{\frac{(s-c)s}{ab}} = \sqrt{(s-a)(s-c)ac} \quad (5)$$

$$a\,AO.\sin\frac{C}{2} = a\sqrt{\frac{(s-a)bc}{s}}\cdot\sqrt{\frac{(s-a)(s-b)}{ab}} = (s-a)\sqrt{\frac{s-b}{s}.ac}. \quad (6)$$

similarly $CO = r\,\mathrm{cosec}\,\frac{A}{2} = \sqrt{\frac{(s-c)ab}{s}}$

$$\cos\frac{A}{2} = \sqrt{\frac{s-(s-a)}{bc}};\quad \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

$$\therefore c.CO.\cos\frac{A}{2} = c\sqrt{\frac{(s-c)ab}{s}}\sqrt{\frac{s(s-a)}{bc}} = \sqrt{(s-a)(s-c)ac}. \quad (7)$$

$$c.CO\sin\frac{A}{2} = c\sqrt{\frac{(s-c)ab}{s}}\sqrt{\frac{(s-b)(s-c)}{bc}} = (s-c)\sqrt{\frac{s-b}{s}.ac}. \quad (8).$$

Hence collecting all the terms and substituting the values of $a.AO\cos\frac{C}{2}$; $a\,AO.\sin\frac{C}{2}$; $c.CO\cos\frac{A}{2}$; $cCO\sin\frac{A}{2}$. found in (5).(6).(7)(8) we $= n(5)$ mehane.

$$a\alpha^2 + b\beta^2 + c\gamma^2 \gtrless a\alpha'^2 + b\beta'^2 + c\gamma'^2$$

$$\left.\begin{aligned} &\sqrt{(s-a)(s-c).ac}\ \cos\overline{X+\frac{\theta}{2}} \\ &+(s-a)\sqrt{\frac{(s-b)ac}{s}}\ \sin\overline{X+\frac{\theta}{2}} \end{aligned}\right\} \gtrless \left\{\begin{aligned} &b\sqrt{\frac{s-b}{s}ac}\ \sin\overline{X+\frac{\theta}{2}} \\ &+\sqrt{(s-a)(s-c)ac}\ \cos\overline{X+\frac{\theta}{2}} \\ &-(s-c)\sqrt{\frac{(s-b)ac}{s}}\ \sin\overline{X+\frac{\theta}{2}} \end{aligned}\right.$$

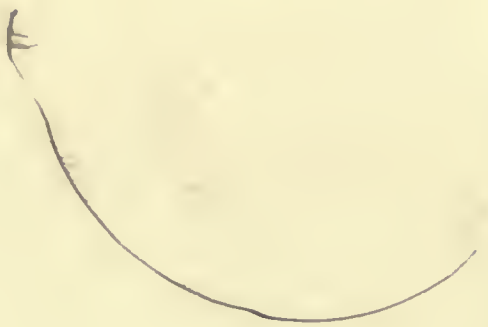
$$(s-a)\sqrt{\frac{(s-b)ac}{s}}\ \sin\overline{X+\frac{\theta}{2}} \gtrless (b-s+c)\sqrt{\frac{(s-b)ac}{s}}.\ \sin\overline{X+\frac{\theta}{2}}$$

$$s-a \gtrless b-s+c.$$

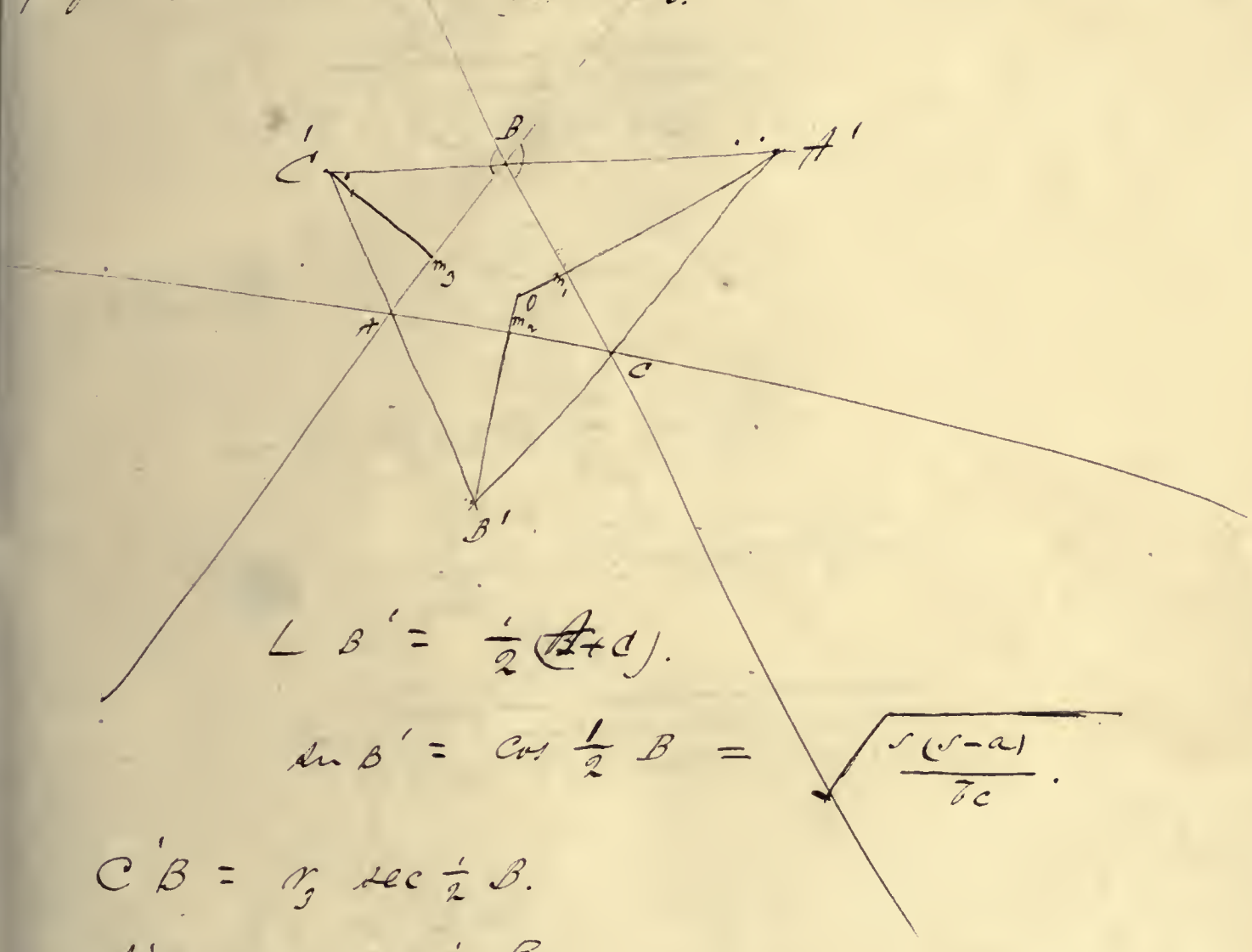
$$2s \gtrless a+b+c.$$

But $2s = a+b+c$ $\therefore a\alpha^2+b\beta^2+c\gamma^2 = a\alpha'^2+b\beta'^2+c\gamma'^2$.

and as the result is independent both of $\theta$ and of the distance $OL$. it is evident the result holds whatever be the value of $\theta$ or whatever be the distance $OL$.



The centres of the escribed ⊙s of a triangle are joined forming another triangle: this is treated similarly and so on; shew that the radii of the circumscribing ⊙s form a geometric progn. whose common ratio is 2.




$$\angle B' = \tfrac{1}{2}(A+C).$$

$$\sin B' = \cos \tfrac{1}{2} B = \sqrt{\frac{s(s-a)}{bc}}.$$

$$C'B = r_3 \sec \tfrac{1}{2} B.$$

$$A'B = r_1 \sec \tfrac{1}{2} B.$$

$$\therefore A'C' = (r_3 + r_1) \sec \tfrac{1}{2} B.$$

$$R_2 = A'O = \tfrac{1}{2} A'C' \sec \tfrac{1}{2} B$$

$$= \tfrac{1}{2} \cdot (r_3 + r_1) \sec^2 \tfrac{1}{2} B.$$

$$r_3 = \frac{\Delta}{s-c} = \sqrt{\frac{s(s-a)(s-b)}{s-c}}. \quad ; \quad r_1 = \frac{\sqrt{s(s-b)(s-c)}}{\sqrt{s-a}}$$

$$\therefore R_2 = \frac{1}{2}\sqrt{s(s-b)}\left\{\frac{\sqrt{s-a}}{\sqrt{s-c}} + \frac{\sqrt{s-c}}{\sqrt{s-a}}\right\}\frac{ac}{s(s-b)}$$

$$= \frac{ac}{2}\,\frac{s-a+s-c}{\sqrt{s.(s-a)(s-b)(s-c)}}$$

$$= \frac{acb}{2\sqrt{s(s-a)(s-b)(s-c)}}.$$

$$\text{Also } R_1 = \frac{b}{2\sin B}.$$

$$\sin B = 2\sin\frac{B}{2}\cos\frac{B}{2}$$

$$= \frac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)}$$

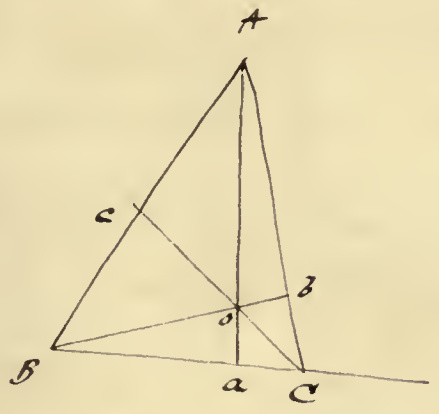
$$\therefore R_1 = \frac{abc}{4\sqrt{s.(s-a)(s-b)(s-c)}}.$$

$$\therefore R_2 = 2R_1$$

Before the successive radii form a geometrical progression with common ratio 2.

⊥ from the ∠ is upon the opposite side meeting a point.

= eqn to AC is. (take a for origin. ∥a A. ac axes of x.)




$$\frac{y}{Aa} + \frac{x}{ac} = 1$$

$$y = -\frac{Aa}{ac}x + Aa.$$

= eqn ⊥ Bb is $y = \frac{Ca}{Aa}(x - Ba)$.

similarly = eqn ⊥ Cc is

$$y = \frac{Ba}{\quad}$$

= eqn to B is $\frac{y}{Aa} - \frac{x}{Ba} = 1$

$$y = \frac{Aa}{Ba}x + Aa.$$

= eqn to Cc is $y = -\frac{Ba}{Aa}(x + aC)$

for the point of intersection of these lines the values of y are equal.

$$\therefore \frac{Ca}{Aa}(x - Ba) = -\frac{Ba}{Aa}(x + aC)$$

$$\frac{Ca}{Aa}x = -\frac{Ba}{Aa}x.$$

or $x = 0$.

∴ the point of intersection is in Aa.

$$\frac{y}{Cc} - \frac{x}{AC} = 1$$

$$y = \frac{Cc}{AC} x + Cc$$

$$y = -\frac{AC}{Cc}(x - CB)$$

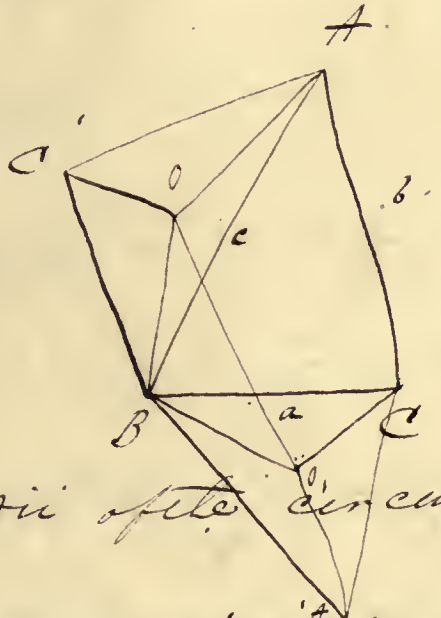
$$0 = -\frac{AC}{Cc}$$

$$y = \frac{CB}{Cc} \cdot (x - AC)$$

$$Cc\,y = -AC\,x + AC \cdot CB$$

$$Cc\,y = CB\,x - CB \cdot CB$$

If on the sides of a $\Delta$ are constructed 3 $\Delta$s whose vertical angles are equal to those of the $\Delta$ respectively opposite shew that the $\Delta$ formed by joining the centres of the $\odot$s circumscribed about them is in every respect equal to the given one.




Let $r, r_1, r_2$ be the radii of the circumscribed $\odot$s

$\therefore r = \frac{c}{\sin C'} = \frac{c}{\sin C}$ for $C' = C$.

$r_1 = \frac{a}{\sin A'} = \frac{a}{\sin A}$ for $A = A' = \frac{c}{\sin C} = r$

$\therefore r = r_1 = r_2$.

$\therefore AO \cdot BO = BO \cdot BO_1$

$\angle OAO = \angle B$ for each $= 2C'$.

$\therefore OO' = AB$

similarly for the other sides.



$$\cos 7\theta = \cos^7\theta - \frac{7.6}{1.2}\cos^5\theta(1-\cos^2\theta) + \frac{7.6.5.4}{1.2.3.4}\cos^3\theta\{1-2\cos^2\theta+\cos^4\theta\}$$

$$+ \frac{7.6.5.4.3.2}{1.2.3.4.5.6}\cos\theta.\{1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta\}.$$

$$= (1+21+35+7)\cos^7\theta - (21+70+21)\cos^5\theta$$

$$+ (35+21)\cos^3\theta - 7\cos\theta.$$

$\therefore \cos 7\theta + 7\cos\theta = 0$ becomes.

$$64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta = 0.$$

$$\therefore \cos^3\theta = 0 \quad \therefore \theta = \frac{\pi}{2} = \frac{3\pi}{2} = \frac{5\pi}{2}.$$

$$8\cos^4\theta - 14\cos^2\theta + 7 = 0.$$

from which the other values of $\theta$ are easily deduced.

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Soluelle = $^{m}$. $\cos 7\theta + 7\cos\theta = 0$.

Hence that $\theta = \sin\theta \sec\frac{\theta}{2} \sec\frac{\theta}{2^2} \sec\frac{\theta}{2^3}$ &c to $\infty$.

For since $\sin\theta = 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}$.

$$= 2^2 \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\sin\frac{\theta}{2^2}$$

$$= 2^3 \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\sin\frac{\theta}{2^3}$$

$$= 2^4 \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cos\frac{\theta}{2^4}\sin\frac{\theta}{2^4}$$

$$= 2^\infty \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3} \text{ &c to infy. } \sin\frac{\theta}{2^\infty}.$$

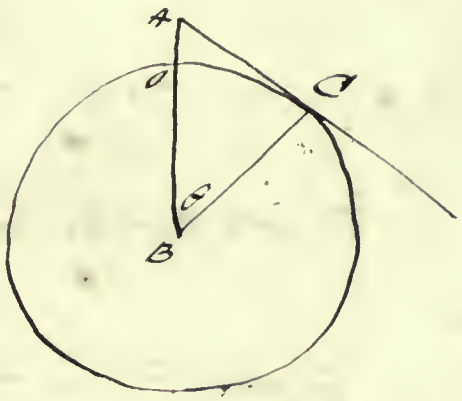
$\therefore 2^\infty \sin\frac{\theta}{2^\infty} = \sin\theta \sec\frac{\theta}{2}\sec\frac{\theta}{2^2}\sec\frac{\theta}{2^3}$ &c to infinity.

But $\sin\frac{\theta}{2^\infty} = \frac{\theta}{2^\infty}$.

$\therefore 2^\infty \sin\frac{\theta}{2^\infty} = \theta$.

or $\theta = \sin\theta \sec\frac{\theta}{2}\sec\frac{\theta}{2^2}\sec\frac{\theta}{2^3}$ &c to infinity.

If the height of a station in feet be increased by $\frac{1}{2}$ and the square root be taken, the result is the distance of the horizon in miles nearly:




$AC = r \tan\theta$.

$\cos\theta = \frac{r}{r+a}$ $\therefore \tan^2\theta = \sec^2\theta - 1$

$= \left(\frac{r+a}{r}\right)^2 - 1$

$\therefore \frac{AC^2}{r^2} = \frac{r^2 + 2ar + a^2 - r^2}{r^2}$

$AC = \sqrt{2ar + a^2} = \sqrt{2ar}$ nearly.

Now $r = 5280 \times 3$ feet nearly.

$= \frac{3}{4} \cdot (5280)^2$. ——

$\therefore AC = 5280 \sqrt{\frac{3}{2}a}$ feet nearly

$= \sqrt{\frac{3}{2}a}$ miles nearly.

$$\frac{\cos A - \sin B \sin C}{\cos B \cos C} = \frac{\cos D - \sin E \sin F}{\cos E \cos F}$$

$$\frac{\cos A - \sin B \sin C}{\cos B \cos C} + 1 = \frac{\cos D - \sin E \sin F}{\cos E \cos F} + 1$$

$$\frac{\cos A - \sin B \sin C + \cos B \cos C}{\cos B \cos C} = \frac{\cos D - \sin E \sin F + \cos E \cos F}{\cos E \cos F}$$

$$\frac{\cos A + \cos(B+C)}{\cos B \cos C} = \frac{\cos D + \cos(E+F)}{\cos E \cos F}$$

$$\frac{\cos B \cos C}{\cos E \cos F} = \frac{-\cos A + \cos B - C}{-\cos D + \cos E - F}$$

$$= \frac{1 - \cos A - (1 - \cos B - C)}{1 - \cos D - (1 - \cos E - F)}$$

$$= \frac{\operatorname{vers} A - \operatorname{vers} B - C}{\operatorname{vers} D - \operatorname{vers} E - F}$$

$$\int^2 \log. \frac{a+\sqrt{a^2-x^2}}{x} = \frac{a^3}{3} \log. \frac{a+\sqrt{a^2-x^2}}{x}$$

$$-\frac{1}{3}\int \frac{a^3}{x}$$

1) $\sin\alpha+\sin\beta+\sin\gamma = 4\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta\cos\frac{1}{2}\gamma.$

$(\alpha+\beta) = 180^\circ-\gamma.$

$\frac{1}{2}(\alpha+\beta) = 90^\circ-\frac{1}{2}\gamma \quad \sin\frac{1}{2}\overline{\alpha+\beta} = \sin 90^\circ-\frac{1}{2}\gamma = \cos\frac{1}{2}\gamma.$

$\sin\alpha+\sin\beta = 2\sin\frac{1}{2}\overline{\alpha+\beta}\cdot\cos\frac{1}{2}\overline{\alpha-\beta}.$

$= 2\cos\frac{1}{2}\gamma\cos\frac{1}{2}\overline{\alpha-\beta}.$

$\sin\gamma = 2\cos\frac{1}{2}\gamma\sin\frac{1}{2}\gamma.$

$\therefore \sin\alpha+\sin\beta+\sin\gamma = 2\cos\frac{1}{2}\gamma\left(\cos\frac{1}{2}\overline{\alpha-\beta}+\sin\frac{1}{2}\gamma\right).$

$= 2\cos\frac{1}{2}\gamma\left(\cos\frac{1}{2}\overline{\alpha-\beta}+\cos\frac{1}{2}\overline{\alpha+\beta}\right.$

$= 4\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta\cos\frac{1}{2}\gamma.$

(2) $\sin\alpha-\sin\beta+\sin\gamma = 4\sin\frac{1}{2}\alpha\cos\frac{1}{2}\beta\sin\frac{1}{2}\gamma.$

$\sin\alpha-\sin\beta = 2\cos\frac{1}{2}\overline{\alpha+\beta}\sin\frac{1}{2}\overline{\alpha-\beta}.$

$= 2\sin\frac{1}{2}\gamma\sin\frac{1}{2}\overline{\alpha-\beta}.$

$\sin\gamma = 2\cos\frac{1}{2}\gamma\sin\frac{1}{2}\gamma.$

$\therefore \sin\alpha-\sin\beta+\sin\gamma = 2\sin\frac{1}{2}\gamma\sin\frac{1}{2}\overline{\alpha-\beta}+2\cos\frac{1}{2}\gamma\sin\frac{1}{2}\gamma.$

$= 2\sin\frac{1}{2}\gamma\left(\sin\frac{1}{2}\overline{\alpha-\beta}+\sin\overline{\alpha+\beta}\right).$

$= 4\sin\frac{1}{2}\gamma\sin\frac{1}{2}\alpha\cos\frac{1}{2}\beta.$

$\sin 2\alpha+\sin 2\beta+\sin 2\gamma = 4\sin\alpha\sin\beta\sin\gamma.$

$\sin 2\alpha+\sin 2\beta = 2\sin\overline{\alpha+\beta}\cdot\cos\overline{\alpha-\beta} = 2\sin\gamma\cos\overline{\alpha+\beta}.$

$\sin 2\gamma = 2\cos\gamma\sin\gamma.$

$\sin 2\alpha+\sin 2\beta+\sin 2\gamma. = 2\sin\gamma\left(\cos\overline{\alpha+\beta}+\cos\overline{\alpha-\beta}\right).$

$= 4\sin\gamma\left(\sin\alpha\sin\beta\right).$

The angles of a spherical Δ are each = 120°. If $s$ be the area of the Δ, $s'$ the area of the polar Δ.

Then $\tan \frac{s}{3} : \tan \frac{s'}{3} :: 6\sqrt{2} + \sqrt{3} : 2\sqrt{2} - \sqrt{3}$.

since each ~~side~~ of the ∠ is of the Δ cont. 120° we have $A + a' = 180°$ ∴ $a' = 60° = B' = C'$.

$$\therefore \frac{a' + b' + c'}{2} = 90°.$$

$$s - a = s - b = s - c = 30°.$$

$$\tan \frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \cdot \sin s - a}} = \frac{1}{\sqrt{2}} \text{ or } \tan A' = 2\sqrt{2}.$$

$$\therefore s' = 3\tan^{-1} 2\sqrt{2} - 180°.$$

$$\frac{s'}{3} = \tan^{-1} 2\sqrt{2} - 60° = \tan^{-1} \frac{2\sqrt{2} - \sqrt{3}}{1 + 2\sqrt{6}}$$

$$\text{also } \frac{s}{3} = 60°.$$

$$\therefore \tan \frac{s}{3} : \tan \frac{s'}{3} :: \sqrt{3} : \frac{2\sqrt{2} - \sqrt{3}}{1 + 2\sqrt{6}}.$$

$$\therefore 6\sqrt{2} + \sqrt{3} : 2\sqrt{2} - \sqrt{3}.$$

Prove that any small error of the Solar Tables in Asc. (R). North Polar distance (Δ), Longitude (I). are connected by the eq.ⁿ

$$\delta\Delta + \sin\Delta \cot l \operatorname{cosec} I\, \delta l + \sin\Delta \cos l \tan I.\ \delta R = 0.$$

Because $\cos\Delta = \sin I . \sin l$

$$\tan R = \cot\Delta \cot I$$

$$\cos l = \cos R \sin\Delta.$$

$$\cos R\, \delta a + \operatorname{cosec}\Delta \cot I\, \delta\Delta + \operatorname{cosec}^2 I \cot\Delta\, \delta I = 0.$$

$$\cos l\, \delta a + \operatorname{cosec}\Delta \cot I\, \delta\Delta + \operatorname{cosec}^2 I \cos\Delta.\ \delta I = 0.$$

$$\text{or } \cot l \sin\Delta \tan I\, \delta R + \delta\Delta + \frac{\cos\Delta \sin\Delta}{\cos I + \sin I}\, \delta I = 0$$

$$\text{or } \delta\Delta + \cos l \sin\Delta \tan I\, \delta R + \sin l \sec I \sin\Delta\, \delta I = 0.$$

The angles of a plane triangle are in geometrical prog. and the common ratio is $\frac{1}{2}$. Shew that the greatest side =

2 perimeter $\sin 12° \frac{6}{7}$.

Let x.y.z. be the sides. $a$. $2a$. $4a$ the angles.

$$\therefore \frac{x}{y} = \frac{\sin a}{\sin 2a} = \frac{1}{2\cos a} \quad \therefore y = 2x \cos a.$$

$$\frac{x}{z} = \frac{\sin a}{\sin 4a} = \frac{1}{4\cos a \cos 2a} \quad \therefore z = 4x \cos a \cos 2a.$$

$\therefore$ if $z$ is the greatest side we have.

$$4\cos a \cos 2a\, 7 = < 2\left(1 + 2\cos a + 4\cos a \cos 2a\right) \sin \frac{a}{2}.$$

$$4\cos\frac{a}{2}\cos a \cos 2a\, 7 = < \left(1 + 2\cos a + 4\cos a \cos 2a\right) 2\sin\frac{a}{2}\cos\frac{a}{2}.$$

$$2\cos\frac{a}{2}\left(\cos 3a + \cos a\right) 7 = < \sin a + 2\sin a\cos a + 4\cos a \sin a \cos 2a.$$

$$\left.\begin{aligned} &\cos\frac{7a}{2} + \cos\frac{5a}{2} \\ &+ \cos\frac{3a}{2} + \cos\frac{a}{2} \end{aligned}\right\} 7 = < \sin a + \sin 2a + \sin 4a.$$

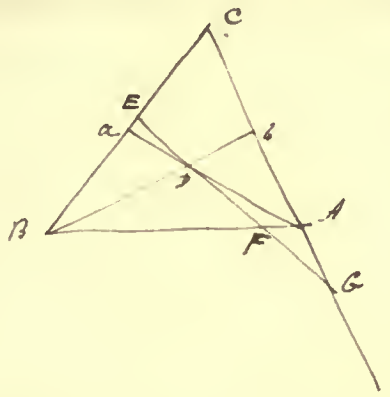
Now since $a + 2a + 4a = 7a = \pi$, $\frac{7a}{2} = \frac{\pi}{2} \therefore \cos\frac{7a}{2} = 0$.

also $\cos\frac{5a}{2} = \sin a$, $\cos\frac{3a}{2} = \sin 2a$. $\cos\frac{a}{2} = \sin 4a$.

$$\left.\begin{aligned} \therefore\ &\cos\frac{7a}{2} + \cos\frac{5a}{2} \\ &+ \cos\frac{3a}{2} + \cos\frac{a}{2} \end{aligned}\right\} = \sin a + \sin 2A + \sin 4a.$$

$\therefore$ the greatest side = 2 perimeter $\sin 12° \frac{6}{7}$.

$$\frac{Aa}{DE} = \frac{3\,Da}{DE} = 3.\frac{\text{sin } CEG}{\text{sin } Aa\,C}$$




$$\therefore \frac{1}{DE} = \frac{3.\text{sin } CEG}{AC.\text{sin } C} = \frac{3}{AC}.\frac{CG}{EG}$$

$$\therefore \frac{GE}{DE} = \frac{3.CG}{AC}$$

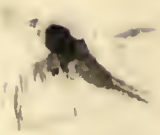
$$\text{Sin}^{a}.\ \frac{1}{DF} = \frac{\text{sin } DFA}{DA.\text{sin } aAB} = 3.\frac{\text{sin } DFA}{DC.\text{sin } B} = \frac{3}{AC}.\frac{\text{sin } DFA}{\text{sin } A} = \frac{3}{AC}.\frac{AG}{FG}$$

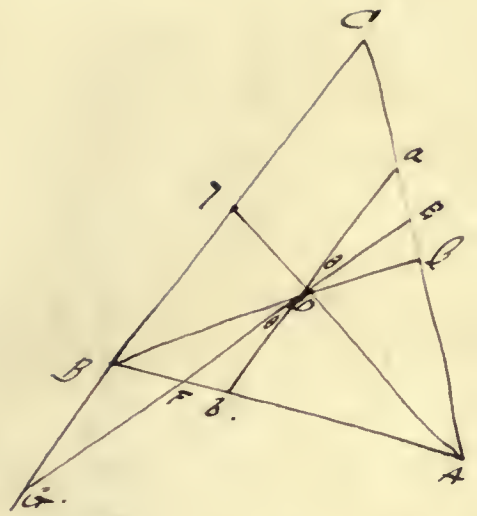
$$\therefore \frac{FG}{DF} = \frac{3.AG}{AC}$$

$$\therefore \frac{GE}{DE} - \frac{FG}{DF} = 3$$

$$\text{or} \quad \frac{DG}{DE} - \frac{DG}{DF} = 1$$

$$\frac{1}{DE} - \frac{1}{DF} = \frac{1}{DG}$$






$$\frac{Da}{DE} = \frac{\sin(C+\theta)}{\sin C} \quad \therefore \quad \frac{1}{DE} = \frac{3}{a}\left\{\frac{\sin(C+\theta)}{\sin C}\right\}.$$

$$\frac{Db}{DF} = \frac{\sin(B-\theta)}{\sin B} \quad \therefore \quad \frac{1}{DF} = \frac{3}{a}\left\{\frac{\sin(B-\theta)}{\sin B}\right\}.$$

$$\therefore \quad \frac{1}{DE} - \frac{1}{DF} = \frac{3}{a}\left\{\cos\theta + \cot C \sin\theta - \cos\theta + \cot B \sin\theta\right\}.$$

$$= \frac{3}{a}\cdot\left\{\frac{\sin(B+C)\sin\theta}{\sin C \sin B}\right\} = \frac{3}{a}\cdot\frac{\sin A \sin\theta}{\sin C \sin B}.$$

$$\frac{Eb}{Db} = \frac{\sin\theta}{\sin(B-\theta)} \quad \therefore \quad Eb = \frac{a}{3}\cdot\frac{\sin\theta}{\sin(B-\theta)} = \frac{b}{3}\cdot\frac{\sin A \sin\theta}{\sin B \sin(B-\theta)} -$$

$$= \frac{c}{3}\cdot\frac{\sin A \sin\theta}{\sin C \sin(B-\theta)}$$

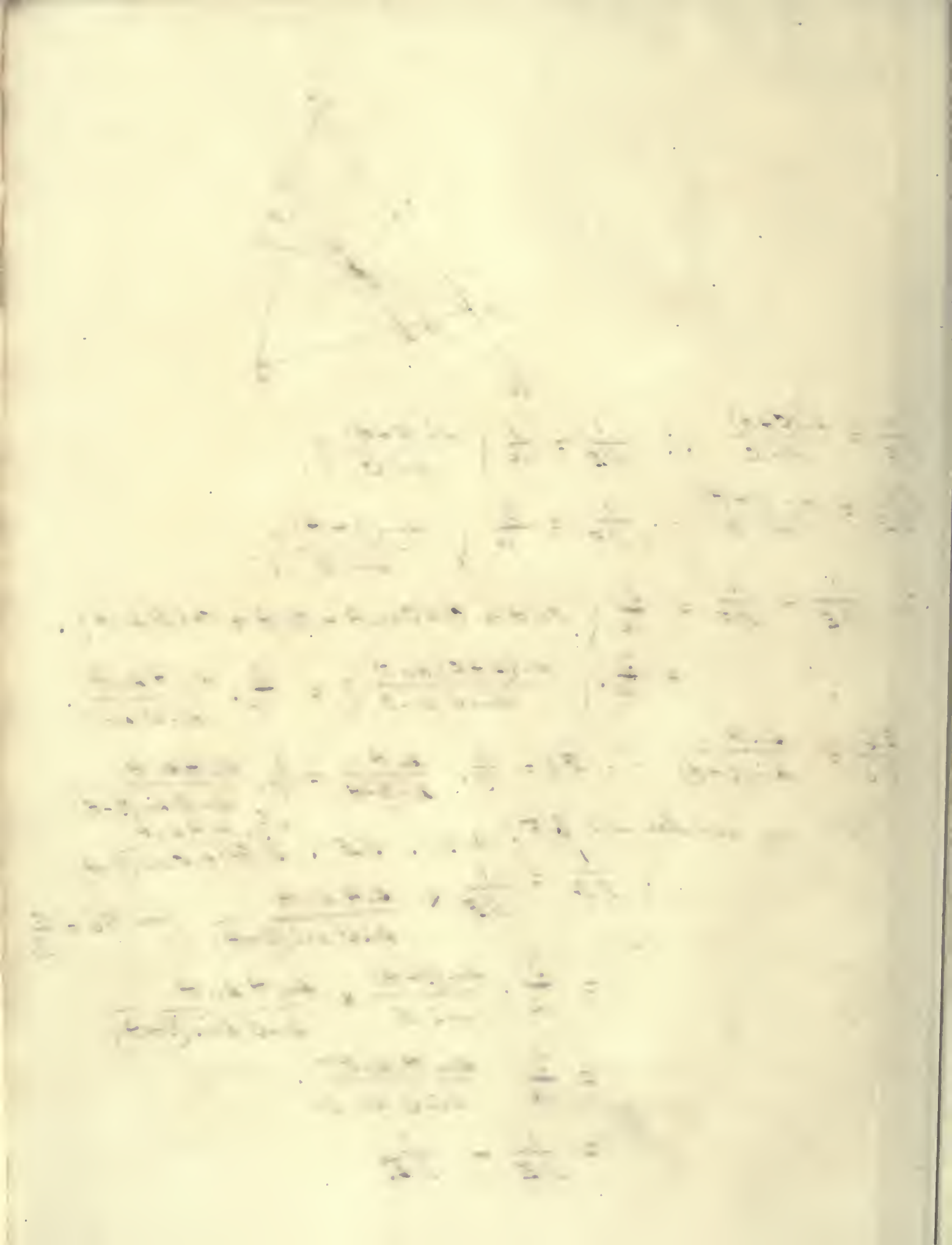
also by similar $\Delta$s $bF : bB :: DE : DG$

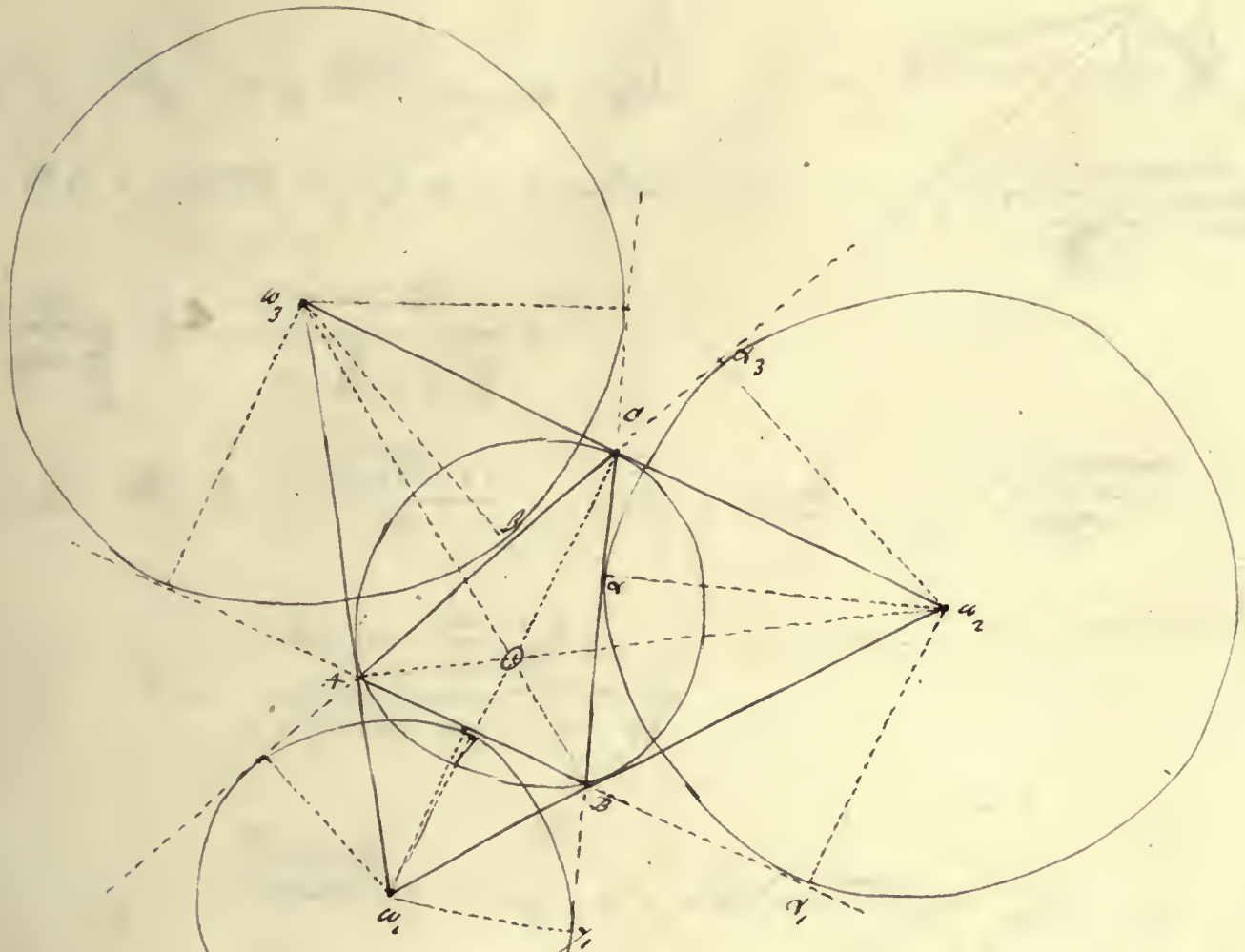
$$\therefore \quad \frac{1}{DG} = \frac{1}{DF} \times \frac{\sin A \sin\theta}{\sin C \sin(B-\theta)} \quad \text{for } Bb = \frac{c}{3}.$$

$$= \frac{3}{a}\cdot\frac{\sin(B-\theta)}{\sin B} \times \frac{\sin A \sin\theta}{\sin C \sin(B-\theta)}$$

$$= \frac{3}{a}\,\frac{\sin A \sin\theta}{\sin C \sin B}$$

$$= \frac{1}{DE} - \frac{1}{DF}$$






(1) $\angle \gamma B \alpha_1 = \angle \alpha B \alpha_1 = \pi - c$

(2). $B\alpha_1 = B\alpha$, $C\alpha_3 = C\alpha$

$\therefore A a_3 + A\alpha_1 = a+b+c = 2s$

and $(A a_3)^2 + (a_3 \alpha_1)^2 = (A a_1)^2 + (a_1 a_1)^2$

$\therefore A a_3 = A a_1 = s$

(3). $\angle \gamma A \omega_1 = \frac{1}{2}(\pi - A)$

$\angle \gamma B \omega_1 = \frac{1}{2}(\pi - B)$

$\therefore \angle A \omega_1 B = \pi - \frac{1}{2}(2\pi - A - B)$

$= \frac{1}{2}(A+B) = \frac{\pi}{2} - \frac{C}{2}$

$\therefore \angle A \omega_1 \gamma = \frac{B}{2}$

$\alpha_1 A (B\omega_2)^2 = (\alpha_1 a_2)^2 + (\alpha B)^2 = (\omega_2 \alpha_1)^2 + (B\alpha_1)^2$.

$\therefore \alpha B = B\alpha_1$

or $B\omega_2$ bisects the angle $\alpha B \alpha_1$

$\omega_1 B \omega_2$ is a straight line.

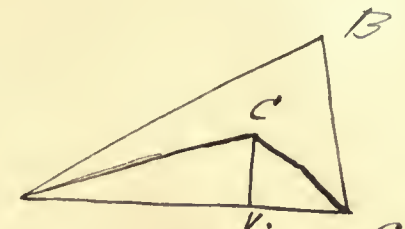
(4) $\angle \omega_1 A \gamma = \frac{\pi}{2} - \frac{A}{2}$.

$\angle OA\gamma = \frac{A}{2}$

$\therefore \omega_1 A \omega_2 = \frac{\pi}{2}$.

(5). $\angle \omega_1 A \gamma = \frac{\pi}{2} - \frac{\pi}{2}$ $\therefore \angle A \omega_1 \gamma = \frac{A}{2}$.






Let $r = AC = r_1$ the reqd. rad.

The. $CV$ = radius of inscribed $\odot$ $= \sqrt{\frac{s(s-a)(s-b)(s-c)}{s}} = r$

$$AC = \frac{r}{\sin\frac{A}{2}} = \frac{r\cos\frac{A}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}}$$

Now $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, and $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

$$\therefore r_1 = \frac{r\cos\frac{A}{2}\cdot 2bc}{\sqrt{s(s-a)(s-b)(s-c)}}$$

$$= \frac{2bc\cos\frac{A}{2}}{s} \quad \text{by substituting for } r.$$

$$= \frac{2bc\cdot\cos\frac{A}{2}}{a+b+c}.$$

C is the centre of the inscribed $\odot$. for $\triangle ABC$.



In any $\Delta$.

$$\cos C = \frac{a^2+b^2-c^2}{2ab}.$$

$$2ab\cos C = a^2+b^2-c^2.$$

If $a$ is constant while $b$ & $c$ vary.

$$2a\cos C\,\delta b = 2b\delta b - 2c\delta c.$$

Similarly we have.

$$2ac\cos B = a^2+c^2-b^2.$$

$$2a\cos B\delta c = 2c\delta c - 2b\delta b.$$

$$\therefore \cos C\,\delta b + \cos B\,\delta c = 0.$$

In a plane Δ. given $a$. $A$. $\overline{b+c}$, or $c$. $A$. $\overline{a+b}$. solve the Δ by the aid of log.s.

In the first case $a+b+c = 2s$ is known.

$$\cos^2\frac{A}{2} = \frac{s(s-a)}{bc}$$

$$\therefore bc = \frac{\cos^2\frac{A}{2}}{s(s-a)}.$$

$$\text{or } \log b + \log c = 2\,\mathrm{I}.\cos\frac{A}{2} - \log s - \log s-a - 2\theta.$$

$$\text{Given } bc. = \beta \text{ suppose}$$

$$\text{also } b+c. = \alpha \text{ suppose.}$$

$$\therefore b^2 + 2bc + c^2 - 4bc = \alpha^2 - 4\beta.$$

$$\therefore b-c = \sqrt{\alpha^2-4\beta}.$$

$$\therefore b = \tfrac{1}{2}\{\alpha + \sqrt{\alpha^2-4\beta}\}.\quad c = \tfrac{1}{2}\{\alpha - \sqrt{\alpha^2-4\beta}\}.$$

$$\text{also } \mathrm{I}.\sin B = \mathrm{I}.\sin A - \log a + \log b.$$

$$\mathrm{I}.\sin C = \mathrm{I}.\sin A - \log a + \log c.$$

In the 2nd case $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$.

$$= \frac{(b+a)(b-a)+c^2}{2bc} = \frac{\beta.\{2b-\beta\}+c^2}{2bc} \quad \text{if } \beta = a+b.$$

$$\therefore (2c\cos A - 2b).b = c^2 - \beta^2.$$

$$b = \frac{c^2-\beta^2}{2\{c\cos A - \beta\}}. \quad \therefore a = \beta - b.$$

and the Δ may be solved as before.

$$\tan^{-1}\frac{4(x-x^3)}{1+x^2} = 2\pi - 2\,\text{chord}^{-1}\frac{2}{\sqrt{1+x^2}}.$$

$$\text{Chord}^{-1}\frac{2}{\sqrt{1+x^2}} = 2\sin^{-1}\frac{1}{\sqrt{1+x^2}}.$$

$$= 2\tan^{-1}\left\{\frac{\frac{1}{\sqrt{1+x^2}}}{\sqrt{\frac{1+x^2-1}{1+x^2}}}\right\} = 2\tan^{-1}\frac{1}{x}$$

$$\therefore \tan^{-1}\frac{4(x-x^3)}{1+x^2} + 4\tan^{-1}\frac{1}{x} = 2\pi.$$

$$\tan^{-1}\frac{4(x-x^3)}{1+x^2} + 2\tan^{-1}\frac{\frac{2}{x}}{1-\frac{1}{x^2}} = 2\pi.$$

$$2\tan^{-1}\frac{2x}{x^2-1} = \tan^{-1}\frac{\frac{4x}{x^2-1}}{1-\frac{4x^2}{x^4-2x^2+1}}$$

$$= \tan^{-1}\frac{4x(x^2-1)}{x^4-6x^2+1}$$

$$\tan^{-1}\frac{4(x-x^3)}{1+x^2} + \tan^{-1}\frac{4x(x^2-1)}{x^4-6x^2+1}$$

$$= \tan^{-1}\frac{4.(x-x^3)\{x^4-6x^2+1-1-x^2\}}{16x^2.(x^2-1)^2 + (1+x^2)(x^4-6x^2+1)} = 2\pi.$$

$$\therefore \text{Lan } 2\pi = 0 = \frac{4(x - x^3)(x^4 - 7x^2)}{16x^2(x^2-1)^2 + (1+x^2)(x^4 - 6x^2 + 1)}$$

$\therefore$ the numerator of this fraction $= 0$.

or $4x^3 \cdot (1-x^2)(x^2-7) = 0$.

$$\therefore \quad x = 0$$

$$x^2 = 1$$

$$x^2 = 7$$

$\log_a m = \log_a b \cdot \log_b c \cdot \log_c d \ldots \log_l m.$

Let $m = a^x = b^y = c^z = d^v \ldots l^x$

$\therefore x = \log_a m \; ; \; y = \log_b m, \; z = \log_c m \ldots$

$a^x = b^y \quad \therefore a = b^{\frac{y}{x}}$

$\therefore \frac{y}{x} = \log_b a.$

or $\dfrac{\log_a m}{\log_a m} = \log_b a \qquad \therefore \log_a m = \log_b m \log_a b.$

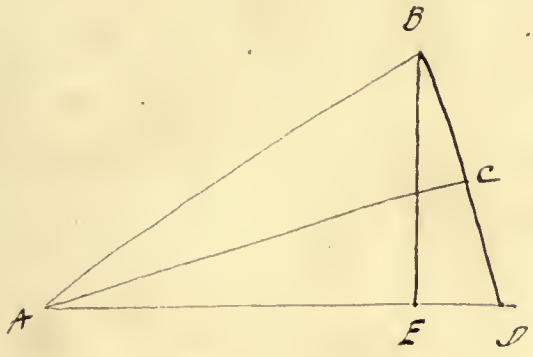
similarly $\log_b m = \log_c m \log_b c.$

$\log_c m = \log_d m \log_c d$

— — —

$\log_k m = \log_l m \log_k l.$

$\therefore \log_a m = \log_a b \log_b c \log_c d \ldots \log_l m.$




$\sin\theta = \frac{BC}{AB}$.

$\therefore \frac{BD^2}{AB^2} = 4\sin^2\theta = \frac{BE^2 + ED^2}{AD^2}$.

$= \frac{BE^2 + (AD - AE)^2}{AD^2}$

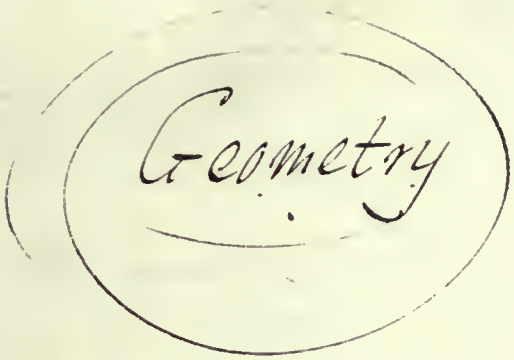
$= \frac{BE^2 + AD^2 - 2\,AD.AE + AE^2}{AD^2}$

$= \frac{2\,AD^2 - 2AD.AE}{AD^2} = 2\left\{1 - \frac{AE}{AD}\right\}$

$\therefore 2\sin^2\theta = 1 - \cos 2\theta$.

$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}}$.

The double value indicated by the radical sign is the 2 values of BC, and DC.


Geometry

Find the eqn to the conoidal surface generated by a horizontal line which meets the helix and the axis of the vertical cylinder on which the helix is traced

Take the axis of the cylinder for the axis of z.

Then $z = \beta$. $y = \alpha x + \gamma$ are the eqns to the line.

and since it always meets the axis of z.

$$0 = 0 + \gamma \quad \therefore \gamma = 0.$$

$\therefore$ the eqns to the generating line are

$$z = \beta. \quad y = \alpha x.$$

Find the generating line meets the helix whose eqns are

$$y = \sqrt{a^2 - x^2}. \quad z = a \cos^{-1} \frac{x}{a}.$$

$$\therefore \alpha x = \sqrt{a^2 - x^2}. \quad (\alpha^2 + 1) x^2 = a^2$$

$$x = \frac{a}{\sqrt{\alpha^2 + 1}}$$

$$\therefore \beta = a \cos^{-1} \frac{1}{\sqrt{\alpha^2 + 1}}.$$

$$\text{or } z = a \cos^{-1} \frac{x}{\sqrt{x^2 + y^2}}$$

which is the required eqn.

Find the area of the $\int_2$ of an ellipsoid cut off by a plane passing thro' the centre, and inclined at $\angle$s $\alpha, \beta, \gamma$ to the axes of $x, y, z$.

The eqn of the plane is $x\cos\alpha + y\cos\beta + z\cos\gamma = 0$.

Hence $\left(\frac{x'\cos\beta - y'\cos\alpha\cos\gamma}{a\sin\gamma}\right)^2 + \left(\frac{x'\cos\alpha + y'\cos\beta\cos\gamma}{b\sin\gamma}\right)^2 + y'^2\frac{\sin^2\gamma}{c^2} = 1$

Now when the eqn to an $\odot$ is $Ax^2 + 2Bxy + Cy^2 = d$, its area is

$$\frac{\pi d}{\sqrt{AC - B^2}}.$$

In this case $AC - B^2$ becomes

$$\frac{1}{\sin^4\gamma}\left(\frac{\cos^2\alpha}{a^2} + \frac{\cos^2\beta}{b^2}\right)\left(\frac{\cos^2\beta}{a^2} + \frac{\cos^2\alpha}{b^2}\right) + \frac{1}{c^2}\left(\frac{\cos^2\beta}{a^2} + \frac{\cos^2\alpha}{b^2}\right)$$

$$- \left(\frac{1}{a^2} - \frac{1}{b^2}\right)^2 \left(\frac{\cos^2\alpha\cos^2\beta\cos^2\gamma}{\sin^4\gamma}\right).$$

$$\frac{\cos^2\gamma}{\sin^4\gamma} \cdot \frac{\cos^4\alpha + \cos^4\beta}{a^2b^2} + \frac{1}{c^2}\left(\frac{\cos^2\beta}{a^2} + \frac{\cos^2\alpha}{b^2}\right).$$

$$+ \frac{2}{a^2b^2}\left(\frac{\cos^2\alpha\cos^2\beta\cos^2\gamma}{\sin^4\gamma}\right\}.$$

$$= \frac{1}{a^2b^2c^2}\left\{a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma\right\}.$$

$$\text{area} = \frac{\pi d}{\sqrt{AC - B^2}}$$

$$= \frac{\pi abc}{\sqrt{a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma}}.$$

Find the equation to the conical surface whose vertex is at the origin and the axis of z, and directrix a circle whose equation is $x^2+y^2+z^2=r^2$, $z=ax+b$.

Take any generating line in any position.

$x=\alpha(z-c)$ (1) $c$ = vertical coordinate of given point.

$y=\beta(z-c)$ (2)

and eliminate $x.y.z.$ between these 4 and

$x^2+y^2+z^2=r^2$ (3) $z=ax+b$ (4).

From (1) and (2) $\frac{x}{\alpha}=\frac{y}{\beta}$ $\therefore \beta x=\alpha y$.

From (1) and (4) $x=\alpha\{ax+b-c\}=\frac{\alpha(b-c)}{1-a\alpha}$.

$$\therefore y=\frac{\beta.(b-c)}{1-a\alpha}.$$

$$z=\frac{\alpha a(b-c)+b-ba\alpha}{1-a\alpha}$$

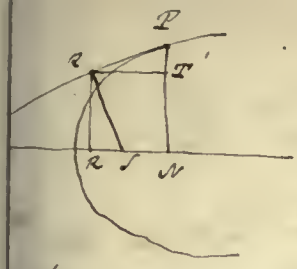
$$=\frac{b-ac\alpha}{1-a\alpha}.$$

Substitute in (3).

$$(\beta^2+\alpha^2)(b-c)^2+(b-ac\alpha)^2=(1-\alpha a)^2r^2.$$

or $\frac{x^2+y^2}{(z-c)^2}.(b-c)^2+\left(b-\frac{acx}{z-c}\right)^2=\left(1-\frac{ax}{z-c}\right)^2r^2.$

$$(x^2+y^2)(b-c)^2+\{b(z-c)-acx\}^2=(z-c-ax)^2r^2.$$




Let $h, k$ be the coord$^{ts}$ of $S$, and let $SN = x$, $NP = y$ be the coord$^{ts}$ of $P$. $PT$ a tangent at that point $SR$ a $\perp$ upon the tangent from $S$. $SR = x'$, $SR = y'$ the co-ords of $R$.

Then from the similar $\triangle s$ $SRR$, $QT'P$

$$\frac{SR}{RR} = \frac{PT'}{QT'} \quad \text{or} \quad \frac{x'}{y'} = \frac{y - y'}{x + x'} \quad \text{or} \quad x'^2 + y'^2 = yy' - xx'$$

also if $\alpha$ be the $\angle$ which the tangent at $P$ cuts the axis of $x$ we have

$$\tan\alpha = \frac{2a}{y} = \sqrt{\frac{a}{x}} = \frac{x'}{y'}$$

$$\therefore \frac{a}{x} = \frac{x'^2}{y'^2} \quad \text{or} \quad x = a\frac{y'^2}{x'^2}.$$

$$y^2 = 4ax = 4a^2\frac{y'^2}{x'^2} \quad \therefore y = 2a\frac{y'}{x'}$$

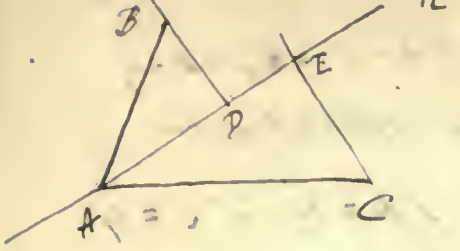
But $x'^2 + y'^2 = yy' - xx'$

$$= 2a\frac{y'^2}{x'} - a\frac{y'^2}{x'}$$

$$= a\frac{y'^2}{x}$$

$$\therefore x'^2 + y'^2 = a\frac{y'^2}{x} \quad \text{the eq}^{\text{n}} \text{ required.}$$

To find the locus of the intersection of the tangent to the parabola, and the $\perp$ upon it from a point $(h, k)$.




Let the forces a & b acting in the direct AB & AC inclined at ∠ 60°

Let AR bisect the ∠ BAC

Now if AB is resolved in the direction AR. DB will be

$$AD = a \cos 30^\circ = \frac{a}{2}\sqrt{3}$$

$$BD = a \sin 30^\circ = \frac{a}{2}.$$

Similarly AC is resolved to AE. EC

$$AE = b \cos 30^\circ = \frac{b}{2}\sqrt{3}$$

$$CE = b \sin 30^\circ = \frac{b}{2}.$$

∴ the resultant in the direct AR =

$$\frac{1}{2}(a+b)\sqrt{3}.$$

Two equal forces acting in direction including 60° find the composition of forces in the line bisecting the angle.

$$\frac{x}{\cos\omega + a} + \frac{y}{a\cos\omega + b} = 1 \qquad \therefore x(a\cos\omega + b) + y(b\cos\omega + a) = (b\cos\omega + a)(a\cos\omega + b)$$

$$\text{or } y = -\frac{a\cos\omega + b}{b\cos\omega + a}x + a\cdot\cos\omega + b$$

$$\therefore m = -\frac{a\cos\omega + b}{b\cos\omega + a}$$

Let $m'$ = the tangent of the angle a tangent the line perpendicular to the given line cuts the axis of $x$

$$\therefore m' = -\frac{1 + m\cos\omega}{m + \cos\omega}$$

$$= -\left\{\frac{1 + \left\{-\frac{a\cos\omega + b}{b\cos\omega + a}\cos\omega\right\}}{-\frac{a\cos\omega + b}{b\cos\omega + a} + \cos\omega}\right\}$$

$$= -\left\{\frac{b\cos\omega + a - a\cos^2\omega - b\cos\omega}{b\cos^2\omega + a\cos\omega - a\cos\omega - b}\right\}$$

$$= -\left\{\frac{a(1 - \cos^2\omega)}{b(\cos^2\omega - 1)}\right\} = \frac{a}{b}$$

$\therefore y = \frac{a}{b}x + $ [illegible] is the equation to the line through the origin parallel to the required line

and since $a$ & $b$ are parts in the same ratio

$$b = \frac{a}{b}a.$$

Subtracting the latter eqn from the former we have

$$y - b = \frac{a}{b}(x - a) \quad \text{or} \quad (y - b)b = (x - a)a$$


North pole star
Cassiopeia
Corona borealis
Arcturus
Cor. Caroli

Let $p$ be the length of the perp — substitute in eq.

$$p = \frac{y' - mx' - c}{\sqrt{1 + 2m\cos\omega + m^2}}$$ , recollecting that $y' = b$, $x' = a$ $m = -\frac{a\cos\omega + b}{b\cos\omega + a}$

$$c = a\cos\omega + b$$

$$p = \frac{\left\{b + \frac{x^2\cos\omega + ab}{b\cos\omega + a} - a\cos\omega - b\right\} d\omega}{\sqrt{1 - \frac{2a\cos\omega + 2b}{b\cos\omega + a}\cos\omega + \left(\frac{a\cos\omega + b}{b\cos\omega + a}\right)^2}} = \frac{\left(a^2\cos\omega + ab - ab\cos^2\omega - a^2\cos\omega\right) d\omega}{\sqrt{(a\cos\omega + b)^2 - 2\cos\omega\{(a\cos\omega + b)(b\cos\omega + a) + (a\cos\omega + b)^2}}$$

$$= \frac{ab\left(1 - \cos^2\omega\right) d\omega}{\sqrt{a^2 + b^2 + 2ab\,(1 - \cos^2\omega}} = \frac{ab\sin^2\omega}{a + b}$$

$$\int dy \int = \frac{a}{\sqrt{a^2-y^2-x^2}}.$$

$$\int_y = a \sin^{-1} \frac{x}{\sqrt{a^2-y^2}} + C.$$

$$\text{from } x = a \cos\alpha = \sqrt{a^2 - a^2\cos^2\alpha - y^2}$$

$$\int_y = a \sin^{-1} \sqrt{\frac{a^2\sin^2\alpha - y^2}{a^2-y^2}}.$$

$$\int = a \int_y^{a\sin\alpha} \sin^{-1} \sqrt{\frac{a\sin^2\alpha - y^2}{a^2-y^2}}$$

$$= \frac{a\pi}{2} \cdot (a - a\cos\alpha).$$

To find the position of the centre of a sphere floating in a fluid.

Let the axes of $X = bY = cZ$.

The equation of the sphere $z = \sqrt{a^2 - x^2 - y^2}$.

The equation of the surface $z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$.

$V = \int_x \int_y \int_z 1 = \int_x \int_y \left\{2\sqrt{a^2 - x^2 - y^2} - c\left(1 - \frac{x}{a} - \frac{y}{b}\right)\right\}$.

$= \int_x \left\{\frac{a^2 - x^2}{2}\pi - \frac{cb}{2}\left(1 - \frac{x}{a}\right)^2\right\}$.

$= \frac{4}{3}\pi a^3 - \frac{cb}{2}\left\{a' - a + \frac{a}{3}\right\} = \frac{1}{6}\left\{8\pi a^3 - cba'\right\}$.

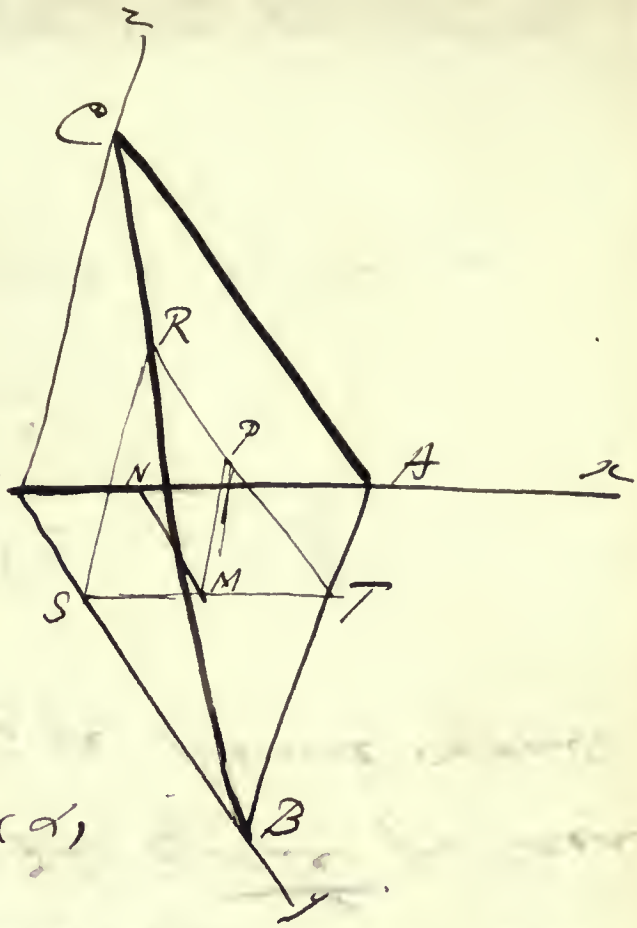
$= \frac{4}{3}\pi a^3 - \frac{cba'}{6}$.

$VX = \int_x \int_y \int_z x = \int_x^a \left\{(a^2 x - x^3)\pi - \frac{cb}{2}\int_0^{a'}\left\{x - \frac{2x^2}{a} + \frac{x^3}{a^2}\right\}\right\}$.

$= a^4\pi - \frac{cba'}{24} = \frac{1}{24}\left\{24a^4\pi - cba'\right\}$.

$VZ = \int_x \int_y \int_z y = \int_x \int_y \left\{2y\sqrt{a^2 - x^2 - y^2} - cy\left(1 - \frac{x}{a} - \frac{y}{b}\right)\right\}$.

$\frac{8}{3}\left(a^2 - x^2\right)^{\frac{3}{2}} - c\left(\frac{y^2}{2} - \frac{xy^2}{a} - \frac{y^3}{6}\right)\Big\}$.




$OA = x \quad AM = y \quad MP = z$

$OA = a \quad OB = b \quad OC = c$

$$\frac{SM}{ST} + \frac{PM}{RS} = 1$$

$$\frac{x}{ST} + \frac{z}{RS} = 1 \qquad (\alpha)$$

$$\frac{ST}{OA} + \frac{OS}{OB} = 1$$

$$\text{or} \quad \frac{ST}{a} + \frac{y}{b} = 1$$

$$\text{Again} \quad \frac{RS}{CO} + \frac{OS}{OB} = 1$$

$$\text{or} \quad \frac{RS}{c} + \frac{y}{b} = 1$$

$$\frac{x}{a\left(1 - \frac{y}{b}\right)} + \frac{z}{c\left(1 - \frac{y}{b}\right)} = 1$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

If a curve in space is a plane curve its plane must coincide with some plane

$$z = Ax + By + C.$$

$$\therefore d_x z = A + B d_x y.$$

$$d_x^2 z = B d_x^2 y.$$

$$d_x \left( \frac{d_x^2 z}{d_x^2 y} \right) = 0.$$

$$\text{or } d_x^2 y \, d_x^3 z - d_x^2 z \, d_x^3 y = 0.$$

and this condtn must be identically satisfied by the coords of the proper curve if the curve is a plane curve.

Prove that the sum of the squares of the projection of any 3 planes at right angles to each other is constant —

no. ellipsoid

$= ax^2 + by^2 + cz^2 + 2a'zy + 2b'xz + 2c'xy + d = 0.$

$\therefore n =$ Let the tangent plane

$(x' - x)\,du_x + (y' - y)\,du_y + (z' - z)\,du_z = 0.$

$du_x = 2ax + 2b'z + 2c'y.$

$du_y = 2by + 2a'z + 2c'x.$

$du_z = 2cz + 2a'y + 2b'x.$

$\therefore (x' - x)\{(ax + b'z + c'y) + (y' - y)(by + a'z + c'x)$

$+ (z' - z)(cz + a'y + b'x) = 0.$

$x'(ax + b'z + c'y) + y'(by + a'z + c'x) + z'(cz + a'y + b'x)$

$- ax^2 - by^2 - cz^2 - b'zx - c'xy - a'zy - c'xy$

$- a'zy - b'xz = 0.$

or $x'(ax + b'z + c'y) + y'(by + a'z + c'x)$

$+ z'(cz + a'y + b'x) + d = 0.$

$\frac{x}{\quad} + \frac{y}{\quad} = 1$

If the tangent plane is // to the axis of z we have.

$$cz + a'y + b'x = 0$$

$$\therefore z = \frac{-(a'y + b'x)}{c}$$

Substitute in the eqn of the surface.

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'xz + 2c'xy + d = 0$$

$$\therefore ax^2 + by^2 + \frac{(a'y + b'x)^2}{c} - 2\frac{(a'y + b'x)^2}{c} + d + 2c'xy = 0$$

$$cax^2 + cby^2 - a'^2y^2 - 2a'b'xy - b'^2x^2 + cd + 2cc'xy = 0.$$

$$(cb - a'^2)y^2 + (ca - b'^2)x^2 + (2cc' - 2a'b')xy + cd = 0.$$

$$\text{or } Ay^2 + Bxy + Cx^2 + cd = 0.$$

The area of this $= \dfrac{\pi c'}{\sqrt{A'B'}}$

where $c' = cd$.

$$A' = \frac{1}{2}\{A + C + \sqrt{A + C + (B^2 - 4AC)}\}.$$

$$B' = \frac{1}{2}(A + C - \sqrt{A + C + B^2 - 4AC}$$

$$\therefore \text{area} = \frac{2\pi cd}{\sqrt{-B^2 + 4AC}}$$

$$= \frac{2\pi cd}{\sqrt{(cb - a'^2)(ca - b'^2) - (cc' - a'b')^2}}$$

If the tangent plane ∥ to the axis of $x$ we have

$$ax + b'z + c'y = 0$$

$$x = -\frac{(b'z + c'y)}{a}.$$

∴ the $=^{n}$ to the projection of the envelope on the plane of $yz$ is

$$\left(\frac{b'z + c'y}{a}\right)^2 + by^2 + cz^2 + 2a'zy = 2\left(\frac{b'z + c'y}{a}\right)^2 + d = 0.$$

$$aby^2 + acz^2 + 2aa'zy - \{b'^2z^2 + 2b'c'zy + c'^2y^2\} + ad = 0.$$

$$(ab - c'^2)y^2 + (ac - b'^2)z^2 + 2(aa' - b'c')zy + ad = 0.$$

The area of this $\odot$ is $\dfrac{\pi ad}{\sqrt{(ab - c'^2)(ac - b'^2) - (aa' - b'c')^2}}$

$$\therefore f_x^2 = \frac{\pi^2 a d^2}{abc - bb'^2 - cc'^2 - aa'^2 + 2a'b'c'}.$$

$$\text{and } f_z^2 = \frac{\pi^2 c d^2}{abc - bb'^2 - cc'^2 - aa'^2 + 2a'b'c'}$$

$$f_y^2 = \frac{\pi^2 b d^2}{abc - bb'^2 - cc'^2 - aa'^2 + 2a'b'c'}.$$

$$\therefore f_x^2 + f_y^2 + f_z^2 = \frac{\pi^2 d^2 \{a + b + c\}}{abc - aa'^2 - bb'^2 - cc'^2 + 2a'b'c'}.$$

If $a.b.c.$ be semi axes of the $\odot$oid.

$$a+b+c = \{a' + b' + c'\} \qquad a'b'c' = abc - aa'^2 - bb'^2 - cc'^2 + 2a'b'c'.$$

Let $u = f(x, y) = 0$ be the Pedal curve.

$\therefore (x' - x)\, du_x + (y' - y)\, du_y = 0$ = the tangent. (1)

The perpendicular from the origin upon this plane

is $y' du_x = x' du_y$. or $y' = \frac{du_y}{du_x} x'$.

$\therefore$ putting for $y'$ in (1) under the form

$$x' du_x + y' du_y = x\, du_x + y\, du_z.$$

and substituting for $y'$

$$x' \left\{ du_x + \frac{(du_y)^2}{du_x} \right\} = x\, du_x + y\, du_z.$$

$$\text{or } x' = \frac{(x\, du_x + y\, du_z)(du_x)}{(du_x)^2 + (du_y)^2}.$$

$$\therefore y' = \frac{(x\, du_x + y\, du_z)(du_y)}{(du_x)^2 + (du_y)^2}$$

$$\therefore p^2 = x'^2 + y'^2 = \frac{\{x\, du_x + y\, du_y\}^2}{(du_x)^2 + (du_y)^2}.$$

$$p = \frac{x\, du_x + y\, du_y}{\sqrt{(du_x)^2 + (du_y)^2}}.$$

... find the area of the ⊙ $ax^2 + bxy + cx^2 + d = 0$. Shewing the necessary condition that it may be real. ⊙

Change the direct of the coord axes. by changing

$$x = x'\cos\theta - y'\sin\theta.$$
$$y = x'\sin\theta + y'\cos\theta.$$

∴ The transformed = n becomes.

$$a\{x'^2\sin^2\theta + 2x'y'\sin\theta\cos\theta + y'^2\cos^2\theta\}$$
$$+ b\{(x'^2 - y'^2)\sin\theta\cos\theta + x'y'(\cos^2\theta - \sin^2\theta)$$
$$+ c\{a'^2\cos^2\theta - 2x'y'\cos\theta\sin\theta + y'^2\sin^2\theta\}.$$

or $(a\cos^2\theta - b\sin\theta\cos\theta + c\sin^2\theta)y'^2 + \{(a-c)2\sin\theta\cos\theta + b\cos^2\theta - \sin^2\theta\}x'y'$
$+ (a\sin^2\theta + b\sin\theta\cos\theta + c\cos^2\theta)x'^2 + d = 0.$

Now make the coeff$^t$ of $x'y' = 0$ we have. $\tan 2\theta = \frac{-b}{a-c}$.

and the = n becomes of the form $Ay'^2 + Bx'^2 = -d$.

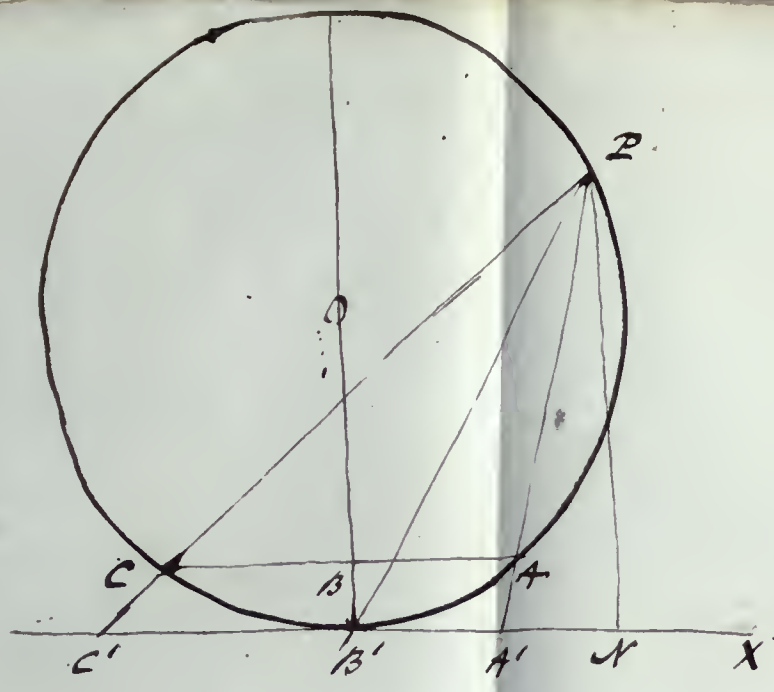
where $A = \frac{1}{2}\{a+c + \sqrt{(a-c)^2 - b^2}\}$ $B = \frac{1}{2}\{a+c - \sqrt{(a-c)^2 - b^2}\}$.

Now area of ⊙ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\pi ab$.

∴ area of ⊙ $\frac{Ay^2}{-d} + \frac{Bx'^2}{-d} = 1$ is $\frac{-\pi d}{\sqrt{AB}}$.

$$= \frac{-2\pi d}{\sqrt{4ac - b^2}}$$

∴ the necessary conditions that it may be an ⊙ are that d must be negative, and $4ac > b^2$.




CPA any segment containing angle CPA. Standing on arc CBA; CB' = B'A then if B'P
be joined, will bisect the angle CPA.

Join CA; and take BC = BA = a. BB' = c., radius of Circle = r, B.O, BX rectangular axes — B' origin — x' = B'N, y' = PN co-ordinates of P.

The circle O is $x^2 = 2ry - y^2$.

$= \frac{y-y'}{x-x'} = \frac{y'-c}{x'+a}$ $\therefore \tan PC'N = \frac{y'-c}{x'+a}$ $\therefore \operatorname{cosec} PC'N = \frac{\sqrt{(y'-c)^2+(x'+a)^2}}{y'-c}$.

$= \frac{\sqrt{2r(c+y')-c^2-y'^2+a^2+2ax'}}{y'-c}$

or $y(a+x') = x'(y'-c) + ay' + cx'$

Let $y = 0$ $\therefore x = B'C' = -\frac{cx'+ay'}{y'-c}$.

$= \frac{y-y'}{x-x'} = \frac{y'-c}{x'-a}$ $\therefore \tan PA'N = \frac{y'-c}{x'-a}$ $\therefore \operatorname{cosec} PA'N = \frac{\sqrt{2r(c+y')-c^2-y'^2+a^2-2ax'}}{y'-c}$.

or $y(x'-a) = x(y'-c) - ay' + cx'$

Let $y = 0$ $\therefore x = BA' = \frac{cx'-ay'}{y'-c}$

Now $C'P = y' \operatorname{cosec} PC'N = \frac{y'}{y'-c}\sqrt{2r(c+y')-c^2+2ax'}$, and $PA' = \frac{y'}{y'-c}\sqrt{2r(c+y')-c^2-a^2}$

$$\therefore \frac{C'P^2}{PA'^2} = \frac{2r(c+y')-c^2+2ax'}{2r(c+y')-c^2-2ax'} = \frac{2cy'-2c^2+2cy'^2+2acx'}{2cy'-2c^2+2cy'^2-2acx'}$$

$$= \frac{c^2(2ry'-y'^2)+y'^2(2rc-c^2)+2acx'y'}{c^2(2ry'-y'^2)+y'^2(2rc-c^2)-2acx'y'} = \frac{c^2x'^2+y'^2a^2+2acx'y'}{c^2x'^2+y'^2a^2-2acx'y'}$$

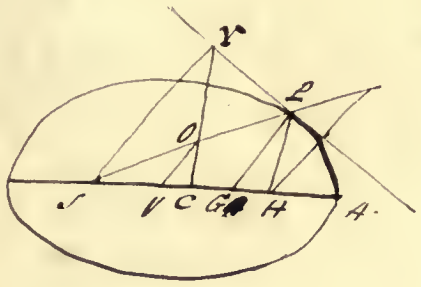
$$= \frac{B'C'^2}{B'A'^2}.$$

or $\frac{C'P}{PA'} = \frac{B'C'}{B'A'}$ $\therefore C'PA$ is bisected by the line $B'P$.



$P$ is a point of an $\odot$, $S$ a focus, $C$ the centre, $PZ$ a tangent at $P$. $SY \perp$ to $PZ$. prove that if $CY$ intersects $SP$ in $O$

$$SO = OP = OZ.$$




Since $SY$ is $\perp$ to the tangent at $P$

$$CZ = CA.$$

& $PG$ be a normal at $G$. $OV \parallel$ to $PG$.

then $CG = e^2x$ $\therefore SG = a(1+e^2x)$

$$\therefore SV = VG = \tfrac{1}{2}a(1+e^2x).$$

$$\therefore VC = \tfrac{1}{2}a(1-e^2x).$$

$\therefore SV : VC :: a+ex : a-ex$

$\therefore CO : OX :: a-ex : a+ex.$

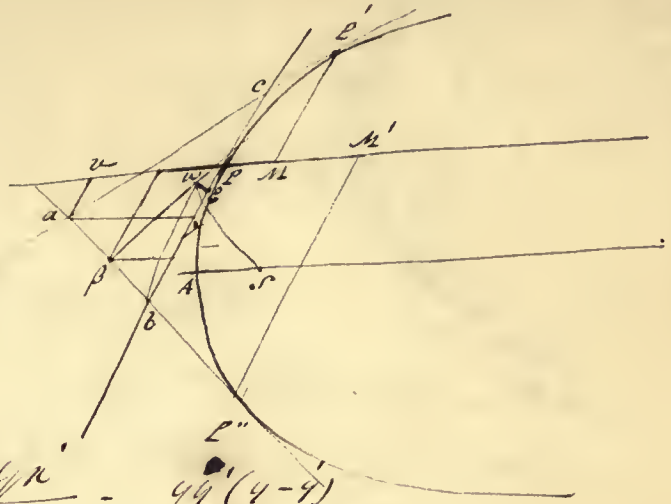
$CO+OX : OZ :: a-ex+a+ex : a+ex.$

$CZ : OZ :: 2a : a+ex.$

But $CZ = a$ $\therefore OZ = \tfrac{1}{2}(a+ex)$.

also $SP = a+ex$ $\therefore SO = OZ = \tfrac{1}{2}(a+ex)$

$\therefore SO = OP = OZ.$




If the ⊙ be referred to the
tangent cb and chord cM' as axes.
cb = h. va = k. the points of intersect.
the diameters. aC'. aC''. PM = x. MC = y
M'C' = x', M'C'' = y' coords. of C'. C''

$\therefore ky = 2a.(x+h)$, $ky' = 2a.(x'+h)$.

$\therefore \frac{x+h}{y}.(x'+h)$, or $h = \frac{xy'-yx'}{y-y'} = \frac{yy'(y-y')}{4a.y-y'}$

$$Pv = \frac{yy'}{4a}. = \frac{yy'}{4a} = \sqrt{xx'}.$$

$$h = \frac{2a\sqrt{x}}{2\sqrt{a}.\sqrt{x}}\left\{\pm\sqrt{x}.+\sqrt{x'}\right\}. = \frac{1}{2}\left\{\pm 2\sqrt{ax} + 2\sqrt{ax'}\right\} = \frac{1}{2}(y'-y) = av.$$

$$\therefore cb = \frac{1}{2}(y+y'); \quad Bb = \frac{1}{2}y' - Py = \frac{1}{2}y = Pc$$
$$yc = Pc + Py = \frac{1}{2}y' = Pb.$$

Let B be the middle point of ab.
The coords of B are $x'' = \frac{1}{2}vP = \frac{1}{2}.\sqrt{xx'}$
$$y'' = \frac{1}{2}\{av + Pb\} = \frac{1}{4}\{2y'-y\}$$

Also since cP = yb. the coords of Q. the middle point of cb
are. $x''' = 0$, $y''' = \frac{1}{2}Py = \frac{1}{4}.(y'-y)$.

Let Pw, Qw, be drawn ⊥ to ab. cb.

The ⊥ to Pw is $y''-Y = m.(x''-X) = \frac{\sqrt{xx'}}{2a}.(x''-X)\frac{\sin\alpha}{\sin\omega-\alpha}$.

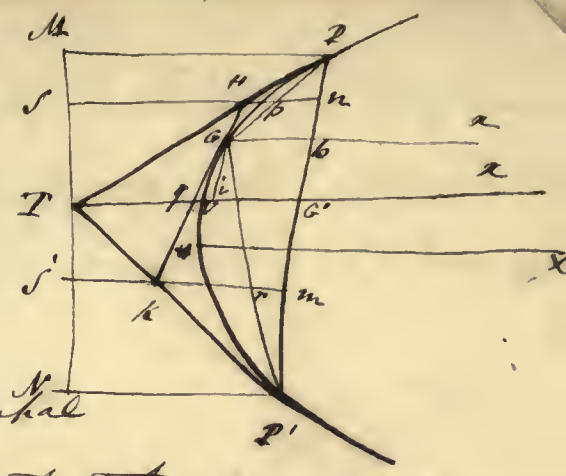
⊥ to Qw is. $y'''-Y = n.X = \frac{\sin\beta}{\sin\omega-\beta}.X$.

where $\cot\alpha = \frac{dy'}{dx'}$ $\cot\beta = -\frac{dy}{dx}$.

Eliminate X and by substituting for x'. y' y''' their values we
get Y, as X. The coords of w. and it will be found that

$$wQ^2 = wP^2$$

Let the tangents at P & P' intersect in T.
& let Hk be a tangent at G, join PG, P'G, PP' — the Δ PGP' shall be double of the Δ HTK.




Through T draw Ta // to AX the principal axis of the parabola. Then PG is an ordinate to the diam. Ta
∴ VT = VG. Similarly if through G the diam. Ga' be drawn, and Gv // to PP' ; vV = qV ∴ Tq = v'G = Gb

Also if Hn, km be drawn // to the axis AX they will bisect GP, GP' in p and r.

and $Hn = \frac{1}{2} PP' = PG'$

Now if MP, NP, &c be drawn // to Tq

area of Δ HTk = $\frac{1}{2}$ Tq × ss'

area of Δ PGP' = $\frac{1}{2}$ Gb × MN

and Tq has been shewn to be = Gb
and MN is double of ss'
∴ Δ HTk is $\frac{1}{2}$ Δ PGP'.

The triangle contained by 2 tangents and S is double of the Δ contd by the 3 Chds. join. the points of contact:

If CL. be any ½ diameter of an ⊙ & AQ be diameter from ends. of the major axis // CL, and meets. the curve Q. and the minor axis produced in O, shew that $2CL^2 = AO.AQ$

Let $x'y'$ be the coords of L.

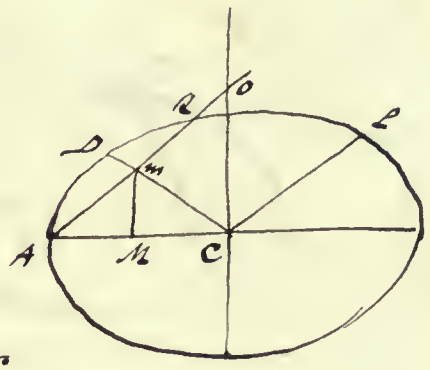
$\therefore y = \frac{y'}{x'}x$ is the = $^{n}$ to CL.

$y = \frac{y'}{x'}(x+a)$ is the = $^{n}$ to AO.

$y = -\frac{b^2x'}{a^2y'}x$ is the = $^{n}$ L CD, the

semi conjugate axis, which bisect AQ in m.

If X. Y. be coords of m. mehane.




$\frac{y'}{x'}(X+a) = -\frac{b^2x'}{a^2y'}X$, or. $\frac{a^2y'^2+b^2x'^2}{a^2y'x'}X = \frac{b^2}{y'x'}X = -\frac{y'}{x'}a.$

$\therefore X = CM = -y'^2\frac{a}{b^2}.$

or $AM = \frac{a}{b^2}\{b^2-y'^2\} = \frac{a}{b^2}\cdot\frac{x^2}{a^2}x'^2 = \frac{x'^2}{a}.$

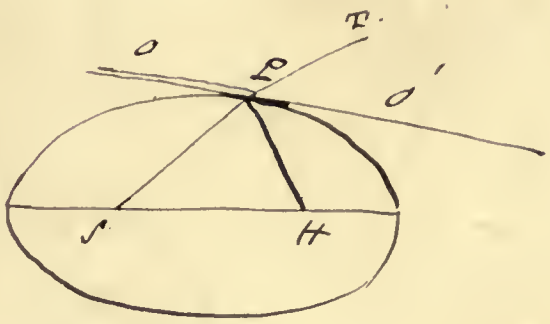
$\therefore AQ = 2AM \text{ sec } QAC = \frac{2x'^2}{a}\sqrt{\frac{x'^2+y'^2}{x'^2}} = \frac{2x'}{a}\cdot\sqrt{x'^2+y'^2}.$

$AO = AC \text{ sec } CAC = \frac{a}{x'}\sqrt{x'^2+y'^2}.$

$\therefore AQ.AO = 2(x'^2+y'^2) = 2CL^2.$



In an elliptic orbit of small eccentricity the Lr velocity is nearly constant about that focus in which the sun is not.




vel in $PO = \frac{h}{SP.\sin SPO}$.

If $PT$ is $\perp$ to $HP$.

vel in $PT = \frac{h}{SP}$.

Lr vel about $H = \frac{h}{SP.PH}$.

which is nearly constant.

# Hydrostatical Problems

acc $\propto \frac{1}{u^4}$. To find the attraction on a particle of fluid in the interior of the fluid, when each particle attracts with a force which varies when the dist. of the particle from the attracted point is finite.

$\therefore$ $xx$ be tangent planes at $O$.

$\therefore$ $Oz$, $oz$ planes of greatest and least curv$^{re}$.

Fig 3.

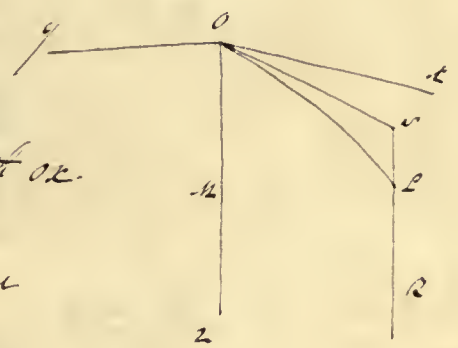
$\therefore$ $Os$ a plane passing thro' $z$ making $\angle\theta$ with $Ox$.

$\therefore$ $Oz$ a normal at $O$, which take for axes of $z$.

$Os = \rho$. $sL = z$. $sR = z_1$, $OM = r$, $ML = u$

$MR = u_1$

The $n^{th}$ to the surface is $z = \frac{\rho^2}{2}\left\{\frac{\cos^2\theta}{R} + \frac{\sin^2\theta}{S}\right\}$.




$u_1 = \rho^2 + (z_1 - r)^2$ ; $u^2 = \rho^2 + (z - r)^2$.

$u_1 \frac{du_1}{z_1} = z_1 - r$ ; $u\,du = \rho - r\,dz$ nearly.

$$= \rho\left\{1 - r\left\{\frac{\cos^2\theta}{R} + \frac{\sin^2\theta}{S}\right\}\right\}.$$

$$\therefore \rho = u\,du\left\{1 + r\left(\frac{\cos^2\theta}{R} + \frac{\sin^2\theta}{S}\right)\right\}.$$

$$\text{Attr}^n = \int_0^{2\pi}\int_0^{\phi}\int_{z_1}^{\rho} \frac{z_1 - r}{u_1}\cdot\frac{1}{u_1^4}.$$

$$\int_{z_1}^{\rho} \frac{z_1 - r}{u_1}\frac{1}{u_1^4} = \int_{z_1}^{\rho}\frac{1}{u_1^4}\frac{du_1}{z_1} = \int_{u_1}\frac{1}{u_1^4} = -\frac{1}{3u_1^3}$$

$z_1 = \rho$ gives $u_1 = \rho$. $z_1 = z$ gives $u_1 = u$.

$$\therefore \int_{z_1}^{\rho}\frac{z_1 - r}{u_1}\frac{1}{u_1^4} = +\frac{1}{3u^3}.$$

$$\text{Attr}^n = \int_0^{2\pi}\int_0^{\phi}\frac{\rho}{3u^3} = \int_0^{2\pi}\int_0^{\phi}\frac{1}{3u^2}\,du\,\rho\left\{1 + r\left(\frac{\cos^2\theta}{R} + \frac{\sin^2\theta}{S}\right)\right\}$$

$$= \int_0^{2\pi}\bullet\left\{1 + r\left(\frac{\cos^2\theta}{R} + \frac{\sin^2\theta}{S}\right)\right\}\cdot\int_u^r\frac{1}{3u^2} = \int_\theta^{2\pi}\left\{\frac{1}{3r} + \frac{1}{3}\left(\frac{\cos^2\theta}{R} + \frac{\sin^2\theta}{S}\right)\right\}$$

$$= \frac{1}{3}\int_\theta^{2\pi}\left\{\frac{1}{r} + \frac{1}{2}\left\{\frac{1+\cos 2\theta}{R} + \frac{1-\cos 2\theta}{S}\right\}\right\} = \frac{2\pi}{3}\left\{\frac{1}{r} + \frac{1}{2}\left\{\frac{1}{R} + \frac{1}{S}\right\}\right\}$$

$\therefore$ Attr$^n$ $= \frac{2\pi}{3}\left\{\frac{1}{r} + \frac{1}{2}\left(\frac{1}{R} + \frac{1}{S}\right)\right\}$ which when $r$ is infinitely small $= \phi$

for the completion of the calculation when the surface of the fluid varies, we have in general.

$$\kappa \cdot \frac{du}{dt}\left(\int_{x=c}^{x=x} dx\right) \frac{1}{2} + \frac{1}{2}\left(1 - \frac{\kappa^2}{X^2}\right)u^2 = g(c-x).$$

Now $X\frac{dx}{dt} = \kappa u \quad \therefore \quad \frac{dx}{dt} = u \cdot \frac{\kappa}{X}$.

$$\frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} = u\frac{du}{dx} \cdot \frac{\kappa}{X}.$$

$$\therefore \frac{\kappa^2}{X}\left(u\frac{du}{dx}\right)N + \frac{1}{2}\left(1 - \frac{\kappa^2}{X^2}\right)u^2 = g \cdot (c-x).$$

$$\left(u \cdot \frac{du}{dx}\right) + \frac{1}{2}u^2 \cdot \frac{X}{N}\left(\frac{1}{\kappa^2} - \frac{1}{X^2}\right) = \frac{g}{\kappa^2} \cdot \frac{X}{N}(c-x).$$

$$e^{\int_x \left(\frac{1}{\kappa^2} - \frac{1}{X^2}\right)\frac{X}{N}} \left\{2u\,du + u^2 \cdot \frac{X}{N}\left(\frac{1}{\kappa^2} - \frac{1}{X^2}\right)\right\} = \frac{2g}{\kappa^2} \cdot \frac{X}{N}(c-x)\, e^{\int_x \left(\frac{1}{\kappa^2} - \frac{1}{X^2}\right)\frac{X}{N}}.$$

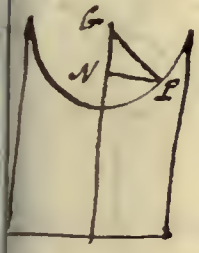
$$d_x \cdot \left\{u^2 \cdot e^{\int_x \left(\frac{1}{\kappa^2} - \frac{1}{X^2}\right)\frac{X}{N}}\right\} = \frac{2g}{\kappa^2}\frac{X}{N}(c-x)\, e^{\int_x \left(\frac{1}{\kappa^2} - \frac{1}{X^2}\right)\frac{X}{N}}$$

Integrating and we have.

$$u^2 \cdot e^{\int_x \left(\frac{1}{\kappa^2} - \frac{1}{X^2}\right)\frac{X}{N}} = \frac{2g}{\kappa^2}\int_x \frac{X}{N}(c-x)\, e^{\int_x \left(\frac{1}{\kappa^2} - \frac{1}{X^2}\right)\frac{X}{N}}$$

$$\therefore \frac{\kappa^2}{2}u^2 = e^{-\int_x \left(\frac{1}{\kappa^2} - \frac{1}{X^2}\right)\frac{X}{N}} \cdot \left\{g\int_x \frac{X}{N}(c-x)\, e^{\int_x \left(\frac{1}{\kappa^2} - \frac{1}{X^2}\right)\frac{X}{N}}\right.$$

A cylindrical vessel cont. fluid is whirled round its axis ⊥ to the horizon with a given angular velocity. find the form which the surface assumes.




Any point in the surf. is acted upon by 2 forces gravity is direct GN — the Centrifugal force is direct NE, and the result. of the fluid is direct EG.

Now the Centrifugal force $= \frac{v^2}{y} = \frac{a^2y^2}{y} = a^2y$

where a = the angular velocity,

$\therefore NE : GN :: a^2y : g$

$y : GN :: a^2y : g$

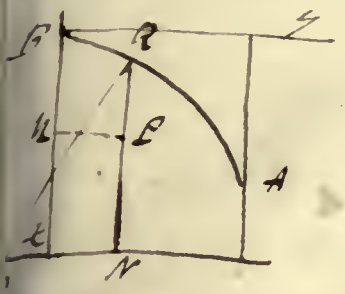
$$GN = \frac{g}{a^2}$$

$\therefore$ GN the subnormal is const.

or the curve a P. where Latus rectum is $\frac{2}{a^2}$.



A fluid is acted on by gravity in direction Oy
and by another in direction Ox which is ∥ to Ox.
Find the ⁿ to the surf.




The forces are g in direction Ay
and μy in direction MP.
Let RE be a normal to the surface
at R. then RE represents the compound
force.

Take B for the origin & y. Bx rectangular axes.

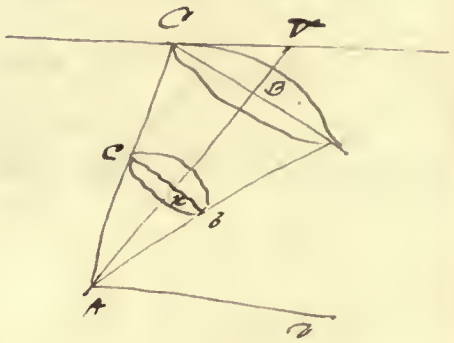
$$\therefore \frac{dx}{y} = \frac{g}{\mu y} \quad \therefore x = \frac{g}{\mu} \log y - \log c$$

$$\frac{\mu x}{g} = \log \frac{y}{c}.$$

$$\therefore \frac{y}{c} = e^{\frac{\mu x}{g}}.$$

$$y = c e^{\frac{\mu x}{g}}.$$

A given cone filled with fluid is supported with its axis inclined to the horizon at a given $\angle$ on which rests. / to the base is this a max.?




Let $\angle BAV = \theta$.

Cb the rect$^{r}$ rect$^{n}$

$AB = a$. $y = mx$ the $= n$ to the generats. A by the revolution of which the cone is generated. $\therefore AC = a \therefore BC = ma$.

The depth of the c. of p. of the rect$^{r}$ rect$^{n}$ below the surf. of the fluid is $ma\cos\theta + (a-x)\sin\theta$.

the area of the rect$^{n}$ is $\pi m^2 x^2$

$\therefore$ the pressure $= \mu\pi m^2 x^2 \{ma\cos\theta + (a-x)\sin\theta\}$.

a maximum.

$\therefore \frac{d}{dx}\{ma\cos\theta + (a-x)\sin\theta\} - x^2\sin\theta = 0$.

$$2a(m\cos\theta + \sin\theta) - 3x\sin\theta = 0.$$

$$x = \frac{2}{3}\left(\frac{am\cos\theta + a\sin\theta}{\sin\theta}\right)$$

$$= \frac{2}{3} AC.$$

4 cubes filled with fluid and made to revolve about an axis thro' the cent. of g. with an angular velocity $\alpha$. Compare the pressure on the sides with the weight of the fluid neglect S. grav.

If $p$ be the pressure on any pt. of the sides we have

$$p = \rho\left\{\tfrac{1}{2}\alpha^2(x^2+y^2)\right\}.$$

Integrate with respect to $x$ from $x = \frac{1}{2}a$ to $x = -\frac{1}{2}a$.

$$p' = \rho\left\{\tfrac{1}{2}\alpha^2\left(\frac{a^3}{3} + ay^2\right)\right\}.$$

Integrate from $y = \frac{1}{2}a$ to $y = -\frac{1}{2}a$.

$$p'' = \rho\left\{\frac{a^4\alpha^2}{3}\right\}.$$

$\therefore$ the pressure on the four sides $= \dfrac{4\rho a^4\alpha^2}{3}$.

also the weight $= gfa^3$

$$\therefore p : w :: \frac{4\rho a^4\alpha^2}{3} : gfa^3$$

$$:: 4\alpha^2 a : 3g.$$

A conical vessel is filled with fluid whose density & depth, the pressure on the base being equal to that on the sides find the vertical $\angle$.

Let $\theta$ = vertical $\angle$.

$a$ = length of the axis of the cone.

$\therefore \pi a^2 \tan^2\theta$ = area of the base.

$\therefore \mu \pi a^4 \tan^2\theta$ = pres on the base.

also $2\pi x a \tan\theta \sec\theta$ = area of small annulus at dist. $x$ measured along the axis

$\therefore 2\pi \mu a^3 x \tan\theta \sec\theta$ = pres on it.

$\therefore \frac{\pi \mu a^4}{2} \tan\theta \sec\theta$ = whole pressure on the sides

$\therefore \frac{\sec\theta}{2} = \tan\theta$ by question.

$\tan^2\theta + 1 = 4\tan^2\theta$

$\frac{1}{\sqrt{3}} = \tan\theta$

or $\theta = 30^\circ$.



If P be a pt. on a cycle. the normal pressure at P = $ds \times x \times \rho$.

where $ds$ is an element of the curve

$\therefore$ the horizontal pressure = $\rho x \frac{dx}{ds} \cdot ds$ if $x = x$.

$= \rho x\, dx$.

similarly the $\perp$ pressure = $\rho x\, dy$.

$\therefore$ the whole horizontal pres = $\frac{1}{2}\rho (x^2 + C)$.

——— vertical ——— = $\rho x (y + C)$.

To determine the nature of the stem of an hydrometer
when the shank (x) of the stem above the surface of
the fluid may be = to the specific grav. of the fluid
in which it is immersed.

Let $a$ = value of $x$ when spec: $g$. $= 1$

$\therefore \frac{x}{a} = s = \frac{w}{u}$.

$\frac{dx}{a} = -\frac{w\,du}{u^2}$.

Let $B$ = whole magnitude of the instrument

$\therefore u = B - \pi \int_x y^2$ (if the stem be a solid frustum).

$\therefore \frac{1}{a} = -\frac{w\pi y^2}{(B - \pi \int_x y^2)^2}$.

~~$B^2 - 2\pi B \int_x$~~

$B - \pi \int_x y^2 = y\sqrt{w\pi a}$.

$-\pi y^2 = dy \cdot \sqrt{w\pi a}$.

$\frac{-\pi}{\sqrt{w\pi a}} = \frac{dy}{y^2}$.

$x\sqrt{\frac{\pi}{wa}} = \frac{1}{y} + C$.

or $yx + C'y = \sqrt{\frac{wa}{\pi}}$.

The = n table $x$ referred both adjustments.

An irregular body is slightly elevated or depressed in a small vessel regd. the time of an oscillation.

Let a. b. be sections of the surface of the fluid and of the plane of flotation.

$dx$ = depression or elevation of the solid

$dy$ = elevation or depression of the water.

Then $dy(a-b) = b.dx$

$\therefore dy = \frac{b\,dx}{a-b}$.

$\therefore dy + dx = \frac{a\,dx}{a-b}$.

$\therefore$ the water displaced $= \frac{ab\,dx}{a-b}$.

$\delta$ = density of the water. $\sigma$ = density of the solid. $V$ = volume of the solid $g$ = force of gravity.

moving force on the solid $= g\delta \cdot \frac{ab.x}{a-b}$.

$\therefore$ accelerating force $= \frac{g\delta ab x}{\sigma V}$.

$\therefore \frac{d^2x}{t} - \frac{g\delta ab}{V\sigma} x = 0$.

$x = A \cos\left\{\sqrt{\frac{g\delta ab}{V\sigma}} \cdot t + B\right\}$.

$x$ is a max when $\cos \sqrt{\frac{g\delta ab}{V\sigma}}\, t = 1$,

$\therefore t = 0, 2\pi, 4\pi$ &c.

wherefore the body will oscillate continually the time of an oscillation being $2\pi\sqrt{\frac{V\sigma}{g\delta ab}}$.

A cylinder which when left to itself floats vertically in the surface of a fluid be elevated through a small depth determine the motion when left to go.

Let $h$ = height of the cylinder.

$\rho$ = sp. grav. $\sigma$ = sp. grav. of the fluid

$a$ = height of elevation.

$x$ = dist of the top of the cylinder from its position at time $t$.

Then the moving force on the cylinder is

$\pi b^2 g f x$. where $b$ = rad of the base.

$\therefore$ accelerating force $= \pi b^2 h \sigma$.

$$\therefore \frac{d^2x}{t} - \frac{g f}{h s} x = 0$$

$$x = A \cos\left(\sqrt{\frac{g f}{h s}}\, t + B\right)$$

$$\frac{dx}{t} = -\sqrt{\frac{g f}{h s}} \cdot A \sin\left(\sqrt{\frac{g f}{h s}} \cdot t + B\right).$$

$x = a$ when $t = 0$.

$$\therefore a = A \cos\left(\sqrt{\frac{g f}{h s}} \cdot 2 + B\right) = A.$$

$\frac{dx}{t} = 0$ when $t = 0$. $\therefore 0 = -\sqrt{\frac{g f}{h s}}\, A \sin B$. $\therefore B = 0$.

$$x = a \cos \sqrt{\frac{g f}{h s}} \cdot t$$

$x$ recedes to its greatest value when $\cos \sqrt{\frac{g f}{h s}} \cdot t = 1$

$$\therefore \sqrt{\frac{g f}{h s}} \cdot t = 2\pi \quad \text{or} \quad t = 2\pi \cdot \sqrt{\frac{h s}{g f}}.$$

which is the time of a complete oscillation

If the body be slightly disturbed, and the $= m$
it will continue to oscillate.

and if the vertical distance of the metacentre
of the body from its c. of grav. $= D.$ and $k^2 = \text{rad}^2$
of gyration, the length of the isochronous
pendulum $= \dfrac{k^2}{D}.$

Ex. a cone floating with its axis vertical.
$h$ = height of the cone. $\therefore \frac{3}{4}h$ = height of its c. of grav.
$\rho$ = density of the fluid $\sigma$ = density of the cone. $a$ = part of its
axis immersed $\therefore \rho a^3 = \sigma h^3 \therefore a = h\sqrt[3]{\frac{\sigma}{\rho}}$.
$\therefore mh\sqrt[3]{\frac{\sigma}{\rho}}$ = rad. of plane of flotation. $\therefore \frac{\pi}{4} \cdot m^4 h^4 \left(\frac{\sigma}{\rho}\right)^{\frac{4}{3}} = Ak^2$
$\therefore \pi m^2 h^2 \cdot \left(\frac{\sigma}{\rho}\right)^{\frac{2}{3}} \cdot \frac{h}{3}\left(\frac{\sigma}{\rho}\right)^{\frac{1}{3}} \cdot AM = \frac{\pi}{4} \cdot m^4 h^4 \left(\frac{\sigma}{\rho}\right)^{\frac{4}{3}}$.

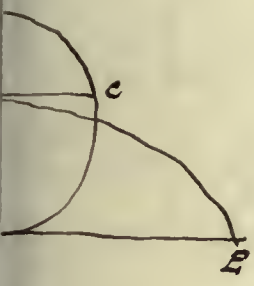
$\therefore AM = \frac{3}{4} \cdot m^2 h \left(\frac{\sigma}{\rho}\right)^{\frac{1}{3}} = \frac{3}{4} m^2 h \sqrt[3]{\frac{\sigma}{\rho}}$

$\therefore$ dist. of the metacentre from the bottom of the cone
is $\frac{3}{4}\left(h\sqrt[3]{\frac{\sigma}{\rho}} + m^2 h\sqrt[3]{\frac{\sigma}{\rho}}\right) = \frac{3}{4}\left(m^2+1\right)h\sqrt[3]{\frac{\sigma}{\rho}}$.

$\therefore$ dist of met. from the c. of g. of the solid $=$

$$\frac{3h}{4}\left\{1 - \left(m^2+1\right)\sqrt[3]{\frac{\sigma}{\rho}}\right\}.$$

$\therefore$ length of the mech. pend $= \dfrac{4k^2}{3h\left(1-\left(m^2+1\right)\sqrt[3]{\frac{\sigma}{\rho}}\right)}$




AB the vertical side of a vessel filled with fluid. D an orifice. DE the course of the issuing stream. BCA a semicircle.

the velocity of the issuing stream = that acquired from B ∴ B is a point in the directrix

$$\therefore AE^2 = 4\,BD \cdot DA$$

$$= 4\,DC^2$$

$$AE = 2\,DC = 2 \sin BC.$$

The times in which two hemispheres are emptied the one by an orifice in the vertex, and the other by an equal orifice in the base are as 3 : 5.
Det. the proport$^{n}$ of the radii.

Let $x$ = rad of the one which is emptied through an orifice in the base.

Then for the motion of the fluid we have.

$$\int \frac{dx}{t} = k\sqrt{2g}x. \quad \text{or} \quad \pi(a^2 - x^2) = k\sqrt{2g}x.$$

$$\frac{\pi}{k\sqrt{2g}}\left\{a^2 x^{-\frac{1}{2}} - x^{\frac{3}{2}}\right) = \int dt \therefore t = \frac{\pi}{k\sqrt{2g}}\left\{2a^2 x^{\frac{1}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right\} + C.$$

Take integral from $x = 0$ to $x = a$.

$$t = \frac{\pi}{k\sqrt{2g}}\left\{\frac{8}{5}a^{\frac{5}{2}}\right\}.$$

Let $a'$ = rad of the other.

$$\therefore \frac{\pi}{k\sqrt{2g}}\left(2a'x^{\frac{1}{2}} - x^{\frac{3}{2}}\right) = \int dt \therefore t = \frac{\pi}{k\sqrt{2g}}\left\{\frac{4}{3}a'x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right\}$$

Take the limits as before $t = \frac{\pi}{k\sqrt{2g}}\left\{\frac{14}{15}a'^{\frac{5}{2}}\right\}$.

$\therefore$ By the question.

$$3 : 5 :: \frac{14}{15}a'^{\frac{5}{2}} : \frac{8}{5}a^{\frac{5}{2}} :: \frac{7}{3}a'^{\frac{5}{2}} : 4a^{\frac{5}{2}}$$

$$12\,a^{\frac{5}{2}} = \frac{35}{3}a'^{\frac{5}{2}}.$$

$$a = a'\left(\frac{35}{36}\right)^{\frac{2}{5}}.$$

Determine the nature of an orifice to be made in the side of a vessel which is always filled to the same altitude, so that the quantity discharged shall be as heights of the portion, and when the whole orifice is open, the quantity discharged = C.

The area of any portion of the orifice = $\int y\,dx$.

$\therefore \int y\,dx\sqrt{2gx} = C\int dx.$

$y\sqrt{2gx} = C = m\sqrt{2gb}$ { $m$ = length of the ordinate at depth $b$ }.

$y^2x = m^2b.$

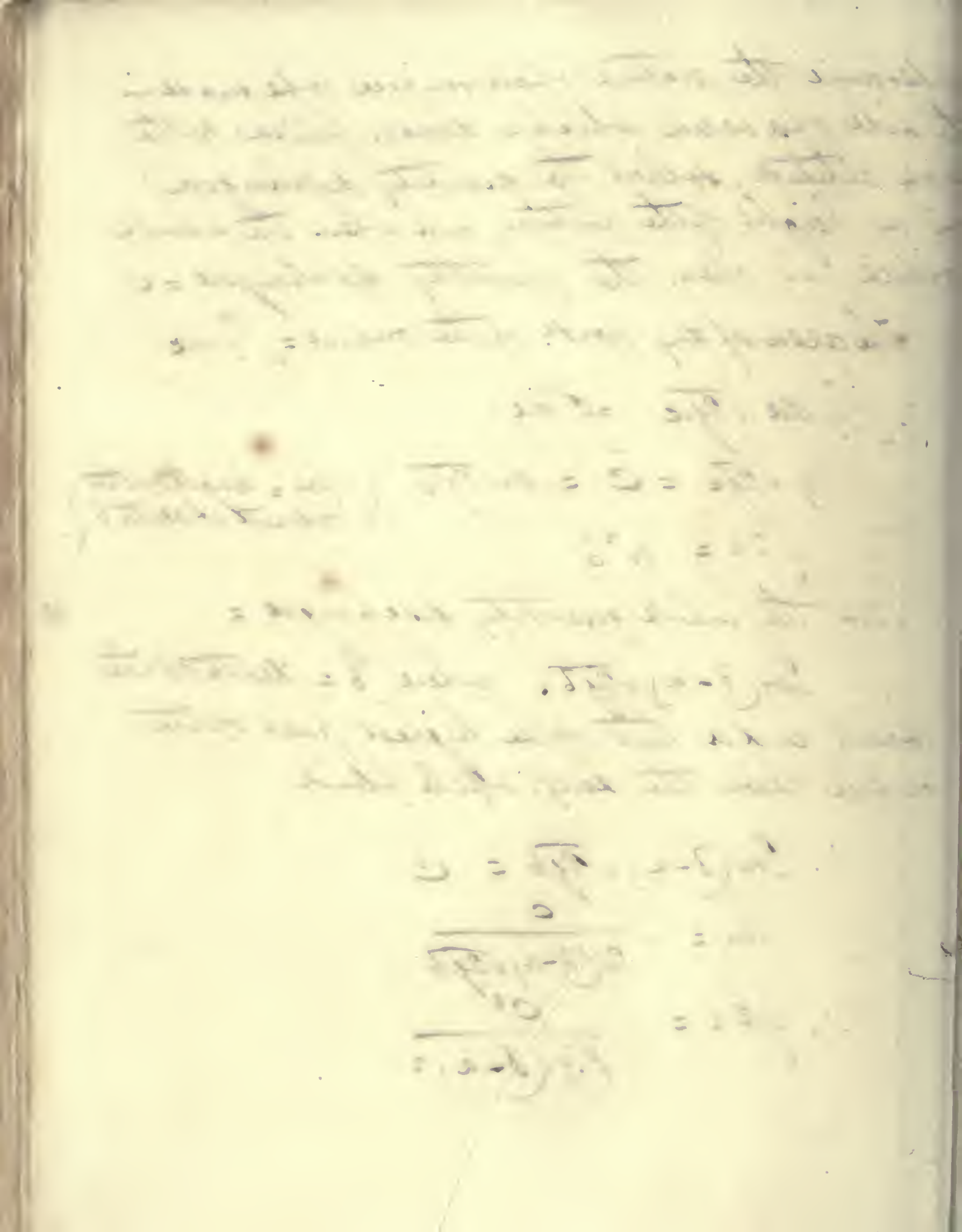
also the whole quantity discharged =

$2m(b-a)\sqrt{2gb}$. where $b$ = depth of the bowels, and $a$ that of the highest part of the orifice below the surf. of the fluid

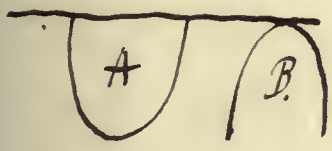
$\therefore 2m(b-a)\sqrt{2gb} = C$

$$m = \frac{C}{2(b-a)\sqrt{2gb}}$$

$$\therefore y^2x = \frac{C^2}{8.g.(b-a)^2}$$



Compare the quantities of fluid discharged by two equal Pipes in the side of a vessel kept constantly full. one of them having its base, and the other its vertex downwards, and the diameters of both coinciding with the surface of the fluid.




Let $y = \sqrt{mx}$ be the curve to B.

$\therefore y = \sqrt{-(mx - ma)} = y = \sqrt{m.(a-x)}$ is the curve to A.

Let the quantities discharged at distances $x$ from the surfaces are.

$2\sqrt{2g}x\sqrt{mx}$, and $2\sqrt{2g}x \cdot \sqrt{m(a-x)}$ respectively.

$\therefore$ the whole quantities discharged are.

$2\sqrt{2g}m \cdot a^2$ and $2\sqrt{2g}m \cdot \int_x \sqrt{ax - x^2}$ or $2\sqrt{2g}m \cdot \frac{a^2\pi}{4}$.

$\therefore$ wherefore $\frac{\text{quantity discharged by A}}{\text{quantity discharged by B}} = \frac{\pi}{4}$.

If a stream of fluid fall ⊥ on a circular plane and also on [illegible] on an elliptic plane whose minor axis equals the diameter of the ○ and impel these planes with equal forces; to determine the proportion of the major and minor axis.

$R \propto A v^2 \sin^2 \theta$.

Put the ○ $A = \pi a^2$

$R \propto v^2 \pi a^2$.

Let $x$ = major axis of the ◯

Then $A = \pi a x$.

$R \propto v^2 \pi a x \sin^2 90^\circ \propto \frac{\pi a x}{4} v^2$

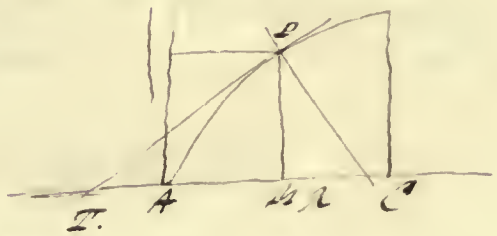
$\therefore \pi a^2 = \frac{\pi a x}{4}$

$x = 4a$.

or the major axis is to the minor as $4 : 1$

Find the resistance to the motion of a solid through a fluid whose velocity is v.

Let R = req'd resistance.




a. y. the coord$^{s}$ of P.

$2\pi y\,ds$ = area of the element described by the revolut$^{n}$ of P about OC.

also $dm^3$ of mom$^{t}$ = $\dfrac{(dy/dx)^3}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}} = \dfrac{\left(\frac{\sqrt{m}}{\sqrt{x}}\right)^3}{\left(\frac{m+x}{x}\right)^{\frac{3}{2}}} = \dfrac{m^{\frac{3}{2}}}{(m+x)^{\frac{3}{2}}}$.

$$\therefore \frac{dR}{v} = \frac{2\pi\sqrt{2m}\,x\,m^{\frac{3}{2}}}{(m+x)^{\frac{3}{2}}}ds = \pi\,2^{\frac{3}{2}}m^2\,\frac{\sqrt{x}}{(m+x)^{\frac{3}{2}}}ds = \pi\,2^{\frac{3}{2}}m^{\frac{1}{2}}\cdot\frac{x}{(m+x)^2}$$

$$R = \pi 2^{\frac{3}{2}}m^2\left\{\int_0^x \frac{m+x}{(m+x)^2} - \int_0^x \frac{m}{(m+x)^2}\right\} = \pi 2^{\frac{3}{2}}m^2\left\{\int_0^x \frac{1}{m+x} - m\int_0^x \frac{1}{(m+x)^2}\right\}$$

$$= \pi 2^{\frac{3}{2}}m^2\left\{\log(m+x) + m\,\frac{1}{m+x}\right\} + C.$$

$R=0$ $x=0$.

$$0 = \pi 2^{\frac{3}{2}}m^2\log m + 1 + C.$$

$$R = 2^{\frac{3}{2}}m^2\pi\left\{\log\left(\frac{m+x}{m}\right) + \frac{m}{m+x} - 1\right\}.$$

If C $x=b$ suppose.

$$\therefore R = 2^{\frac{3}{2}}m^2\pi\left\{\log\left(\frac{m+b}{m}\right) + \frac{m}{m+b} - 1\right\}.$$

$\sqrt{x}\cdot\sqrt{2ax} = 2a$

Compare the resistance on a cube moving in the direction of its diagonal with that on a cube moving in a direction ⊥ to one of its sides.

Let $v$ = velocity. $a$ = side of the stream tube. The density of the fluid is $\frac{1}{\sqrt{3}}$.

$\therefore$ if $R$ = resist on the cube moved diagonally.

$r$ = ——————————— ⊥ to a side

$R : r :: \frac{1}{3\sqrt{3}} \times 3 : 1$

$\therefore R = \frac{r}{\sqrt{3}}$

$r = R\sqrt{3}$.

Find the Resistance of a Stream Line motion when it is the solid generated by the revolution of a Cycloid about its axis.

Let R = Resistance.

$$\therefore \frac{dR}{dx} = 2\pi y \cdot \frac{\left(\frac{dy}{dx}\right)^3}{1+\left(\frac{dy}{dx}\right)^2} \times \tfrac{1}{2}\rho v^2$$

$$= 2\pi \left\{ a \operatorname{vers}^{-1}\frac{x}{a} + \sqrt{2ax-x^2} \right\} \cdot \frac{(2a-x)^{\frac{3}{2}}}{2a\sqrt{x}} \Big\} \tfrac{1}{2}\rho v^2$$

$$\therefore R = \frac{\pi\rho v^2}{2a} \int_x \left( a \operatorname{vers}^{-1}\frac{x}{a} \right) \left\{ \frac{(2a-x)}{\sqrt{x}}^{\frac{3}{2}} \right\} + \frac{\pi\rho v^2}{2a} \int_x (2a-x)^2.$$

$$\text{Now} \int_x \frac{(2a-x)^{\frac{3}{2}}}{\sqrt{x}} = \int_x \frac{(2a-x)^2}{\sqrt{2ax-x^2}} = \int_x \frac{2a^2}{\sqrt{2ax-x^2}} + 2a\int_x \frac{a-x}{\sqrt{2a-x}} - \int_x \sqrt{2ax-x^2}$$

$$= 2a^2 \operatorname{vers}^{-1}\frac{x}{a} + 2a\sqrt{2ax-x^2} - \frac{x-a}{2}\sqrt{2ax-x^2} - \frac{a^2}{2}\operatorname{vers}^{-1}\frac{x}{a}.$$

$$\int_x \frac{(2a-x)^{\frac{3}{2}}}{\sqrt{x}} a \operatorname{vers}^{-1}\frac{x}{a} = a \operatorname{vers}^{-1}\frac{x}{a} \left\{ \frac{3a^2}{2}\operatorname{vers}^{-1}\frac{x}{a} + \frac{5a-x}{2}\sqrt{2ax-x^2} \right\}.$$

$$-a\int_x \left\{ \frac{3a^2}{2}\operatorname{vers}^{-1}\frac{x}{a} \frac{1}{\sqrt{2ax-x^2}} + \frac{5a-x}{2} \right\}.$$

$$= \frac{3a^3}{2}\left(\operatorname{vers}^{-1}\frac{x}{a}\right)^2 + a\operatorname{vers}^{-1}\frac{x}{a}\left(\frac{5a-x}{2}\sqrt{2ax-x^2}\right) -$$

$$- \frac{3a^3}{4}\left(\operatorname{vers}^{-1}\frac{x}{a}\right)^2 - \frac{1}{2}\left\{ 5a^2x - \frac{ax^2}{2} \right\} + C.$$

$$= \frac{3a^3\pi^2}{4} - 4a^3 = \frac{1}{4}\left\{ 3a^3\pi^2 - 16a^3 \right\} \quad \text{from } x = 2a \text{ to } x = 0.$$

# Optical Problems




ABD is the course of the axis of a pencil which is incident directly upon the first surface of a prism. find its primary and secondary foci after refraction.

Let $BR = u$ $BC = a$. $\iota$ = inclination of the prism $Aq_2 = v_2$ $Aq_1 = v_1$ the primary secondary and primary foci. $R_1$ the geometrical focus of the pencil after refraction at the first surface.

Then $AB = a \sin \iota = z$.

$$BR_1 = \mu u \quad \therefore AR_1 = \mu u + a \sin \iota$$

If $\phi$, $\phi'$ be angles of incidence and emergence at A.

$$Aq_1 = v_1 = \frac{\mu \cos^2 \phi'}{\cos^2 \phi} (\mu u + a \sin \iota)$$

$$Aq_2 = v_2 = \mu (\mu u + a \sin \iota)$$

If the incident ray not direct at B. but $\psi$. $\psi'$ ∠ of refraction & reflection. Then $BR_2 = \mu u$, $BR_1 = \frac{\mu \cos^2 \psi'}{\cos^2 \psi} u$



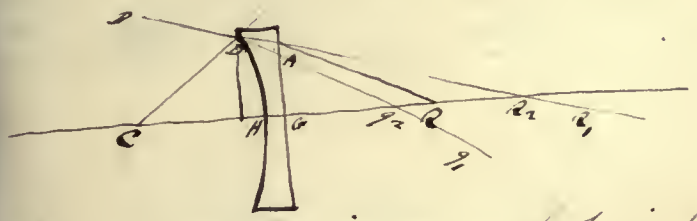

$$\therefore AR_2 = a \frac{\sin \iota'}{\cos(\psi' + \iota)} + \mu u.$$

$$AR_1 = a \cdot \frac{\sin \iota}{\cos(\psi' + \iota)} + \frac{\mu \cos^2 \psi'}{\cos^2 \psi} u$$

$$\therefore \frac{\mu \cos^2 \phi'}{\cos^2 \phi} v_1 = \frac{a \sin \iota}{\cos(\psi' + \iota)} + \frac{\mu \cos^2 \psi'}{\cos^2 \psi} u$$

$$v = a \frac{\cos^2 \phi}{} \frac{\sin \iota}{} + \frac{\cos^2 \psi' \cos^2 \phi}{} u$$

A pencil is incident obliquely on a plano concave lens. find the foci of the emergent pencil




D the origin of the pencil. $q_2$. $q_1$ the foci after refr: at the first surf.
$q_2$ $q_1$ ——— second surf.

$AR = u$ $Dq_2 = v_2$ ; $Dq_1 = v_1$ $CH = r$. $CH = r$ the radius of the $\odot$ of which $DH$ is an arc. $\varphi\ \varphi'$ $\angle$s of incidence and refract: at $A$.
$\psi\ \psi'$ $\angle$s of emergence and incidence at $R$.

To find $AD$. let $CH = x$. $DH = y$.

$$y = \sqrt{r^2 - x^2} = (r + t + GR_2 - x)\tan\varphi.$$

$$r^2 - x^2 = (r + t + GR_2)^2 - 2(r + t + GR_2)x + x^2.$$

$$\therefore x = \frac{(r + t + GR_2)^2 - r^2}{2(r + t + GR)}$$

$$AR_2 = \mu u \therefore GR_2 = \mu u \cos\varphi \quad \therefore x = \frac{(r + t + \mu u\cos\varphi)^2 - r^2}{2(r + t + \mu u\cos\varphi)}$$

$$AR_1 = \frac{\mu\cos^2\varphi'}{\cos^2\varphi}\, u.$$

$$\therefore DR_2 = \mu u + \frac{(r + t + \mu u\cos\varphi)^2 - r^2}{2(r + t + \mu u\cos\varphi)}.$$

$$DR_1 = \frac{\mu\cos^2\varphi'}{\cos^2\varphi}\, u + \frac{(r + t + \mu u\cos\varphi)^2 - r^2}{2(r + t + \mu u\cos\varphi)}$$

$$\text{Now } \frac{\mu\cos^2\psi'}{v_1} - \frac{\cos^2\psi}{DR_1} = -\mu\frac{\cos\psi' - \cos\psi}{r}$$

$$\text{and } \frac{\mu}{v_2} - \frac{1}{DR_2} = -\mu\frac{\cos\psi' - \cos\psi}{r}.$$

Calculate the posn and dimensions of the caustic C of alternate rays of a pencil refracted at a plane surface. The origin of the pencil being at a dist. $a$ from the surface, and the distance of the extreme ray from the axis $s$.




C the origin E the geometrical focus. Hr. kE the direct. of the coherent rays after refract. kr. Ir the direct. of any other rays. Meeting Hs in t.

Let the rad. of the caustic C of alternate $AK = y'$ $AH = y$. $\mu = \frac{4}{3}$.

$$\frac{\text{incr}}{\text{inc}} = \frac{Hr}{HE} = \frac{Hr}{y'} \quad \therefore \text{incr} = \text{inc} \cdot Hr \cdot \frac{1}{y'},$$

similarly — $\text{ins} = \text{inc} \cdot Hr \cdot \frac{1}{y}$. $\therefore sr = \text{inc} \cdot Hr \left(\frac{y+y'}{yy'}\right)$.

But $Es : Es :: y'^2 : y^2 \therefore rs : Es :: y^2 - y'^2 : y^2$

$$\therefore rs = Es \cdot \frac{y^2 - y'^2}{y^2} = \text{inc} \cdot Hr \cdot \frac{y+y'}{yy'} \quad \text{or} \quad \text{inc} = \frac{Es}{Hr} \cdot \left(\frac{y-y'}{y'}\right) \cdot \frac{y'}{y}.$$

If $y'$ be such that inc is a minimum. $y = 2y' \therefore y' = \frac{y}{2}$.

$$\text{inc} = \frac{y}{4} \frac{Es}{Hr}. \qquad \text{incr} = \frac{1}{4} Es$$

Now $Hr = \mu u + \frac{\mu^2 - 1}{\mu} \cdot \frac{y'^2}{2u}. = \frac{4.20}{3} + \frac{7.3}{9.4} \cdot \frac{25}{4.2.20}$

$$= \frac{80}{3} + \frac{35}{384}. = \frac{10275}{384}.$$

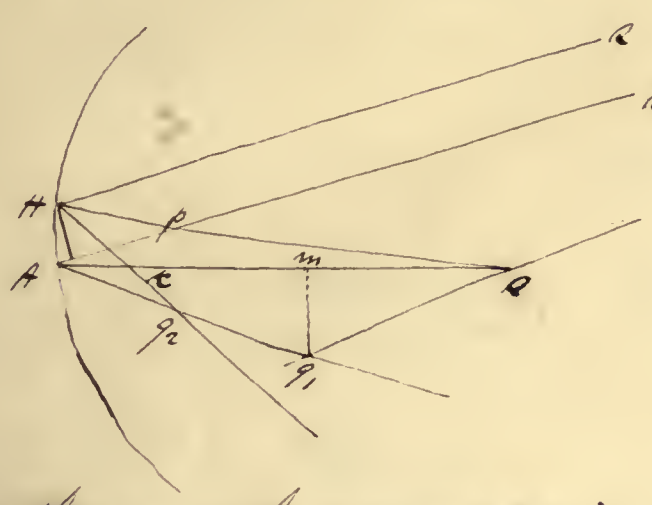
$$Es = \frac{\mu^2 - 1}{\mu^3} \cdot \frac{y^2}{2u}$$

$$= \frac{7.3}{9.4} \cdot \frac{25}{40} = \frac{7.5}{12.8} = \frac{35}{96}.$$

$$\therefore Hm = \frac{10275 + 140}{384} = \frac{10415}{384} = \frac{}{}$$

$$\text{inc} = \frac{5}{4} \cdot \frac{35}{96} \cdot \frac{384}{10275} = \frac{35}{2055} = \frac{7}{411}$$

A pencil of // rays is incident obliquely at an ∠ of 30° on a spherical reflecting surface whose radius is 40 inches. find the focal lines of the emergent pencil




CA the axis of the incident pencil. $Aq_1$ the axis of the reflected pencil — since the pencil consists of // rays. the line thro' O. from the origin may be considered // to CA.

$$\therefore \angle q_1OA = \angle q_1HO = 30°.$$

$$\therefore Aq_1 = \frac{r}{2}\sec 30° = \frac{20.2}{\sqrt{3}} = \frac{40}{\sqrt{3}}.$$

axes of $q_2$ the ∠ of incidence of RA. R'H.

$\theta+\delta\psi$ — ∠ of reflect. of R'H.

$$\delta\psi = \angle R'HO - RAO = \alpha PO - \alpha CO = HOA = \frac{AH}{r} \text{ nearly.}$$

$$\delta\psi = q_2HO - q_2AO = \psi - HRA - \psi + Hq_2A$$

$$= \frac{AH\cos\varphi}{q_2} - \frac{AH}{r} = \frac{AH}{r}$$

$$\therefore \frac{1}{q_2} = \frac{2}{r\cos\varphi} = \frac{2}{20\sqrt{3}} = \frac{1}{10\sqrt{3}}$$

$$\therefore q_2 = 10\sqrt{3}. = \frac{30}{\sqrt{3}}.$$

A small pencil is inclined obliquely on a plane refracting surface. The distance of the origin from the point of incidence is 20 inches, and the refractive index $\frac{4}{3}$. Find the position of the focal lines. The angle of incidence being 30°.




Let $AQ = 20$.

$HQ_2 = v_2$ } the secondary
$HQ_1 = v_1$ } and primary foci.

Then $\frac{\mu \cos^2\phi_1'}{v_1} - \frac{\cos^2\phi}{u} = 0$.

$$\therefore v_1 = \frac{\mu u \cos^2\phi'}{\cos^2\phi} = \frac{u}{\cos^2\phi}\{\mu(1-\sin^2\phi')\}$$

$$= \frac{u}{\cos^2\phi}\{\mu - \sin^2\phi\}.$$

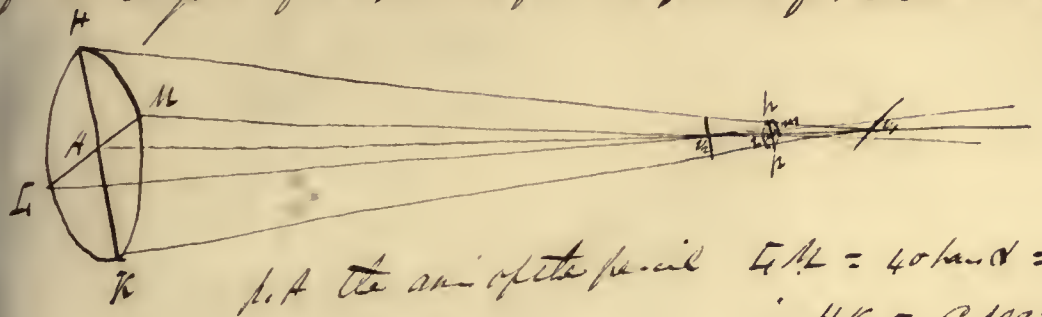
$$= \frac{20.4}{3}.\left\{\frac{4}{3} - \frac{1}{4}\right\}$$

$$= \frac{20.4.13}{3.3.4} = \frac{260}{9}$$

also $v_2 = \mu u = \frac{4.20}{3} = \frac{80}{3} = \frac{240}{9}$.

$$\therefore v_1 - v_2 = \frac{20}{9}.$$

Find the position and dimensions of the C of least confusion of an oblique pencil incident on a plane surface at an ∠ of 30° after refraction. The refractive index being $\frac{4}{3}$, and the distance of the origin of the pencil from the point of incidence 20 inches, semi-vertical ∠ = α.




Let the axis of the pencil $LM = 40 \tan\alpha = c$ suppose

$\therefore HK = c \sec 30° = \frac{2c}{\sqrt{3}}$.

Now $\frac{hk}{\mu\mu_1} = \frac{HK}{A\mu_1} = \frac{2c}{\sqrt{3}.v_1}$ or $hk = \left(\frac{v_1 - 40}{v_1}\right)\frac{2c}{\sqrt{3}}$

$\frac{lm}{Mv_2} = \frac{LM}{Av_2} = \frac{c}{v_2} \therefore lm = \left(\frac{40 - v_2}{v_2}\right)c$.

But $lm = hk \therefore \frac{40 - v_2}{v_2} = \frac{2(v_1 - 40)}{\sqrt{3}.v_1}$ or $40(\sqrt{3}.v_1 - 2v_2) = v_1 v_2(\sqrt{3}+2)$

$$40 = \frac{v_1 v_2(\sqrt{3}+2)}{v_1\sqrt{3} - 2v_2}.$$

$$\therefore lm = \left(\frac{v_1(\sqrt{3}+2)}{v_1\sqrt{3} - 2v_2} - 1\right)c$$

$$= \frac{2(v_1 + v_2)c}{v_1\sqrt{3} - 2v_2}.$$

But $\mu\frac{\cos^2\phi'}{v_1} - \frac{\cos^2\phi}{u} = 0$ $\quad \frac{1}{v_1} = \frac{\cos^2\phi}{\mu u \cos^2\phi'} = \frac{3.3}{4.4.20.\left(1 - \frac{9}{4.16}\right)}$

$$= \frac{3.3.4.\cancel{4}}{4.4.\cancel{4}.5} = \frac{9}{27.5}$$

$\therefore v_1 = \frac{275}{9} = 30\frac{5}{9}$.

$v_2 = \mu u = \frac{4.20}{3} = \frac{80}{3} = 26\frac{2}{3}$.

$$\therefore lm = \frac{2\left\{.5\frac{1}{5}\right\}c}{30\frac{.5\sqrt{3}}{9} - \frac{480}{9}} = \frac{1030c}{150\sqrt{3} - 480}$$

A spherical surface

Find the position of the focal lines of an oblique pencil reflected at a convex spherical surface.




Let O the centre of the surface & the axis the pencil incident & reduced backward & meet AO in $q_2$ the secondary foci, & any ray which after reflection is produced backward to meet $Aq_2$ in $q_1$ the primary focus. ∴ if this be considered as the pencil incident in direction more nearly opposite to that of the incident pencil, both $Aq_1$ $Aq_2$ are $-ve$.

$\angle HAA = \frac{HH}{HA} = \frac{AH \cdot HH}{HA} = \frac{AH \cos\varphi}{u}$ where. $\{\varphi$ - diff $\angle$ of inc. rad. of $H, HA$

$\text{now } \varphi = \angle RAL - \angle RHL'$

$= \angle ROA + \angle ORA - \angle ROH - \angle ORH$

$= \angle HOA + \angle HRA = \frac{AH}{r} + \frac{AH \cos\varphi}{u}.$

$\text{also } \varphi = q_1HO - q_1HO = K - HOA - K + Hq_1A$

$= \frac{AH \cos\varphi}{v_1} - \frac{AH}{r}.$

$\therefore \frac{\cos\varphi}{v_1} - \frac{1}{r} = \frac{\cos\varphi}{u} + \frac{1}{r}.$

$\frac{\cos\varphi}{-v_1} + \frac{1}{u} = -\frac{2\cos i}{r\cos\varphi}$

Let $\angle AOq_2 = \theta$ $\therefore -\frac{AO}{Aq_2} = \frac{-\sin(\varphi+\theta)}{\sin\theta} \cdot \frac{AO}{AR} = \frac{\sin(\varphi-\theta)}{\sin\theta} : -u$

$\therefore \frac{1}{-v_2} + \frac{1}{u} = -\frac{2\cos\varphi}{r}.$

Plane refraction at a curved spherical surface.




$$\delta\varphi = \angle\, QAP - \angle\, QHP = \frac{AH}{r} + \frac{AH\cos\varphi}{u}$$

$$\delta\varphi' = \angle\, q_1AP - \angle\, q_1HP'$$

$$= \{AO'O + O'OA\} - (Hq_2O + q_2OH)$$

$$= \{\overline{Hq_2O} + O'q_1q_2 + O'OA\} - (\overline{Hq_2O} + q_2OH)$$

$$= Hq_1A + HOA = \frac{AH\cos\varphi'}{v} + \frac{AH}{r}$$

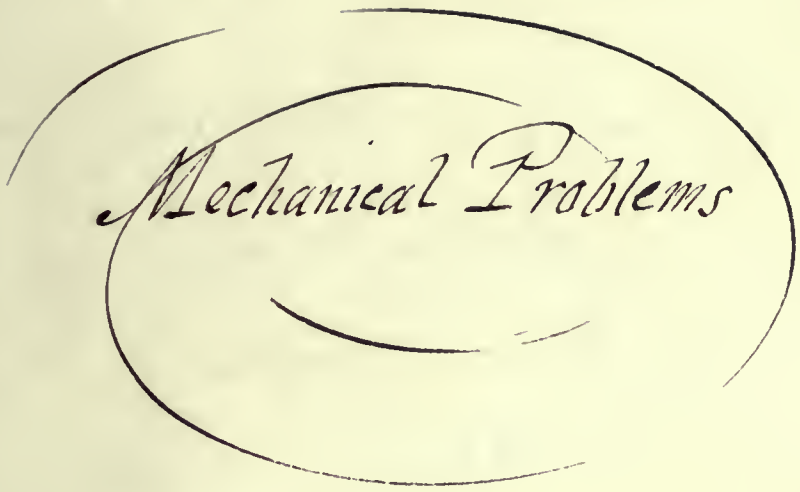
$$\therefore \frac{\delta\varphi}{\delta\varphi_1} = \frac{\mu\cos\varphi'}{\cos\varphi} = \frac{\frac{1}{r} + \frac{\cos\varphi}{u}}{\frac{1}{r} + \frac{\cos\varphi'}{v}}$$

$$\therefore \frac{\mu\cos^2\varphi}{v} - \frac{\cos^2\varphi}{u} = -\left\{\frac{\mu\cos\varphi' - \cos\varphi}{r}\right\}$$

$$\frac{-r}{AQ} = \frac{\sin(\varphi - \theta)}{\sin\theta} = -\sin\varphi\cot\theta + \cos\varphi$$

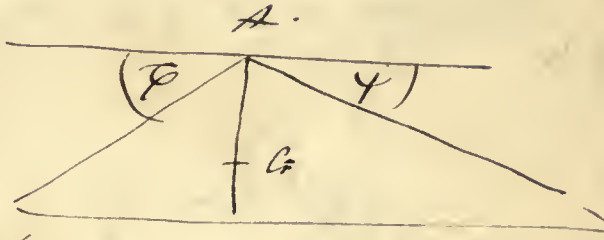
$$\frac{-r}{AO} = \frac{\sin(\varphi' - \theta)}{\sin\theta} = -\sin\varphi'\cot\theta + \cos\varphi'$$

$$\therefore \frac{\mu}{v_2} - \frac{1}{u} = -\frac{1}{r}\left\{\mu\cos\varphi' - \cos\varphi\right\}.$$


Mechanical Problems

Silent Exam. of Prob. The 2 arms all $\underline{\alpha}$ $\underline{2B}$. and the angle makes small oscillations in one plane. The length of the isochronous pendulum is

$$\frac{2}{3}\cdot\frac{a^3+b^3}{\sqrt{a^4+2a^2b^2\cos\theta+b^4}}$$




$AG = h$.

Now $M(k^2+h^2) = \frac{1}{3}(a^3+b^3)$.

$$\frac{1}{2}Mh = a^2\sin\varphi + b^2\sin\psi \quad (1)$$

$$0 = a^2\cos\varphi - b^2\sin\psi \quad (2)$$

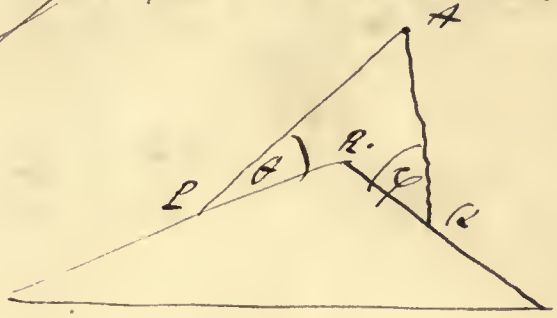
Square (1) and (2), and add.

$$\left(\frac{1}{2}Mh\right)^2 = a^4 + 2a^2b^2\cos\theta + b^4$$

$\therefore$ Length of isochronous pendulum =

$$\frac{2}{3}\cdot\frac{a^3+b^3}{\sqrt{a^4+2a^2b^2\cos\theta+b^4}}.$$

Two weights are connected by a string passing round a fixed pulley. and slide along two inclined planes. shew that the velocities of the two weights at any time are inversely as the cosines of the angles which their strings make with the planes.




$RL = s.\ RR = s'$

Now. $\delta AR = \delta s'. \cos\phi$

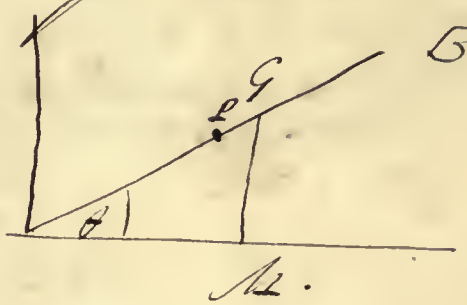
$\delta AL = \delta s \cos\theta$ } ultly.

$\therefore \delta AR = \delta AL.$

$\therefore \delta s' \cos\phi = \delta s. \cos\theta.$

$$\frac{\frac{ds}{t}}{\frac{ds'}{t}} = \frac{\cos\phi}{\cos\theta}.$$

And revolves about one extremity in a horizontal plane: find the diffl. eqn of the path of a ring which slides along it.




$AL = r \cdot \angle ACG = \theta$.

$AL = r \cdot AG = a$.

$k$ = mass of rod. $m$ = mass of ring.

Constant of areas.

$$M\left(\left(k^2+a^2\right)\frac{d\theta}{t} + mr^2\frac{d\theta}{t}\right) = C \quad (1)$$

Vis Viva

$$M\cdot\left(k^2+a^2\right)\left(\frac{d\theta}{t}\right)^2 + m\left(\left(\frac{dr}{t}\right)^2 + r^2\left(\frac{d\theta}{t}\right)^2\right) = C_1$$

$$\text{Now } \frac{d\theta}{t} = \frac{C}{M\left(k^2+a^2\right)+mr^2}.$$

$$\left\{\frac{M\left(k^2+a^2\right) + m\left(\frac{dr}{t}\right)^2 + \left(\frac{dr}{\theta}\right)^2}{M\left(k^2+a^2\right)+mr^2}\right\} = k^2$$

the diffl. eqn required.

Nicholsons hydrometer whose weight is $w$, is slightly depressed the weight of the fluid displaced becomes $w+w'$: when the hydrometer is riding reaches its posn: of $=^{n}$ a small weight $w''$ is placed in the upper dish: det. the motion, and shew that if an additional weight $= \sqrt{w'^2+w''^2+\frac{w''}{w}w'^2}$. be inserted at a proper time, the instrument may be reduced to a state of $=^{m}$

Let $a$ be the depth of the cofgr. of the instrument from its posn: of rest at the time the weight of the fluid displaced is $w+w$

$x$ = depth of the same at a time $t$.

$\therefore$ for the motn. $\frac{d^2x}{t} = -\frac{g}{a}.g.\frac{w'}{w} = \frac{g}{a}.\frac{w'}{w}x.$

$$\therefore \frac{d^2x}{t} = -\frac{g}{a}.\frac{w'}{w}x^2 + C.$$

but $\frac{dx}{t} = v$ when $x = a$.

$$\therefore \frac{d^2x}{t} = \frac{g}{a}\frac{w'}{w}.(a^2-x^2).$$

At the posn: of $=^{m}$. $x=0$ $\therefore \frac{dx}{t} = a\sqrt{\frac{g}{a}.\frac{w'}{w}}.$

Let the weight $w''$ be now placed in the upper dish.

$\therefore \frac{d^2x'}{t} = -\frac{g}{a}.\frac{w'}{w+w''}.x$ is the $=^{n}$ for the subsequent motn.

$$\therefore \left(\frac{dx'}{t}\right)^2 = -\frac{g}{a}.\frac{w'}{w+w''}x'^2 + C$$

when $x'=v$. $\left(\frac{dx'}{t}\right)^2 = \frac{g}{a}.\frac{w'}{w}a^2$

$$\therefore \left(\frac{dx'}{t}\right)^2 = \frac{g}{a}.\frac{w'}{w+w''}\left\{\frac{w+w''}{w}.a^2 - x'^2\right\}.$$

$$\frac{dt}{dx'} = \sqrt{\frac{(w+w'')a}{w'g}}.\frac{1}{\sqrt{\frac{w+w''}{w}a^2-x'^2}}$$

$$\therefore t = \sqrt{\frac{(w+w'')a}{w'g}}.\sin^{-1}\frac{x'\sqrt{w}}{a\sqrt{w+w''}} + C$$

$t=0$ $x'=a$. $\therefore C=0$.

$$t = \sqrt{\frac{(w+w'')a}{w'g}}.\sin^{-1}x'\sqrt{\frac{w}{w+w''}}.$$

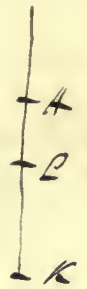
# Nicholsons Hydrometer.

A the surface of the fluid.

B the plane of flotation.

C L = x. Then for the motion of fluid if

L = d.




$$\frac{d^2x}{t} = -\frac{gx}{d}.$$

$$\left(\frac{dx}{t}\right)^2 = C - \frac{gx^2}{d} \quad \text{When } x = d\frac{w'}{w},\ \frac{dx}{t} = 0.$$

$$\overline{\frac{dx}{t}}^2 = \frac{g}{d}\left\{d^2\left(\frac{w'}{w}\right)^2 - x^2\right\}$$

and $(vel)^2$ when it passes thro' the position of $= v^2$

$$= g d \left(\frac{w'}{w}\right)^2.$$

If fluid subsequent motion, measuring x from the new plane of flotation which is below the former by a quantity $= d\frac{w''}{w_1}$, we have

$$\frac{w+w''}{g}\left(\frac{dx}{t}\right)^2 = C - \frac{wx^2}{d}.$$

and when $x = d\frac{w''}{w_1}$ we have

$$\overline{vel}^2 = g d \left(\frac{w'}{w}\right)^2.$$

Nicholsons hydrometer.

$$\frac{w+w''}{g}\cdot\left(\frac{dv}{t}\right)^2 = (w+w'')\,d\left(\frac{w'}{w}\right)^2 + w\,d\left(\frac{w'}{w}\right)^2 = \frac{w\,x^2}{d}.$$

and when $\left(\frac{dv}{t}\right) = 0$.

$$\frac{w\,x}{d} = (w+w'')\,d\left(\frac{w'}{w}\right)^2 + w\,d\left(\frac{w''}{w}\right)^2$$

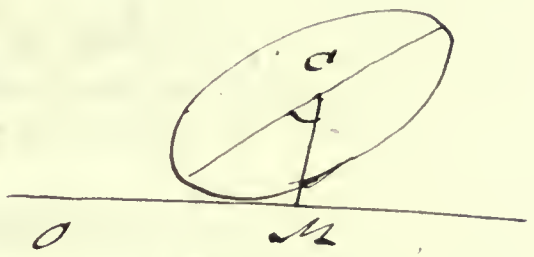
and additional weight to be placed to sustain it at the additional depth $= w\cdot\frac{x}{d}$

$$\therefore \frac{w'x^2}{d^2} = (w+w'')\,\frac{w'^2}{w^2} + w''^2.$$

$$\therefore \text{Additional weight} = \sqrt{w'^2 + w''^2 + \frac{w''}{w}\,w'^2}.$$

A rotatory motion is given to an elliptic cylinder which is then suddenly laid lengthwise upon a smooth horizontal plane: det. the subsequent motion. What must be the original velocity that the body may assume a position of unstable rest?

Let $v_0$ be the initial velocity, immediately after the cylinder is placed on the plane.




$CM = y$; $M$ = its mass.

Then $M\left\{\left(\frac{dy}{dt}\right)^2 + k^2\left(\frac{d\phi}{dt}\right)^2\right\} = 2Mg\{h - y\}$.

$$y = a\sqrt{1 - e^2\sin^2\phi} = \sqrt{a^2\cos^2\phi + b^2\sin^2\phi}.$$

$$\frac{dy}{dt} = -\frac{a^2\cos\phi\sin\phi + b^2\sin\phi\cos\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}\cdot\frac{d\phi}{dt}.$$

$$= -\frac{(a^2 - b^2)\sin\phi\cos\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}\frac{d\phi}{dt}.$$

$$\left(\frac{d\phi}{dt}\right)^2 \left\{ k^2 + \frac{(a^2-b^2)\sin^2\phi\cos^2\phi}{a^2\cos^2\phi + b^2\sin^2\phi} \right\}$$

$$= C - 2g\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}.$$

Let $\alpha$ be the original value of $\phi$.

$$\therefore \omega^2 \left\{ \frac{a^2+b^2}{4} + \frac{(a^2-b^2)\sin^2\alpha\cos^2\alpha}{a^2\cos^2\alpha + b^2\sin^2\alpha} \right\} = C - 2g\sqrt{a^2\cos^2\alpha - b^2\sin^2\alpha}$$

$$\therefore \left(\frac{d\phi}{dt}\right)^2 \left\{ \frac{a^2+b^2}{4} + \frac{(a^2-b^2)\sin^2\phi\cos^2\phi}{a^2\cos^2\phi + b^2\sin^2\phi} \right\}$$

$$- \omega^2 \left\{ \frac{a^2+b^2}{4} + \frac{(a^2-b^2)\sin^2\alpha\cos^2\alpha}{a^2\cos^2\alpha + b^2\sin^2\alpha} \right\}$$

$$= 2g \left\{ \sqrt{a^2\cos^2\phi + b^2\sin^2\phi} + \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha} \right\}.$$

If it assume a point of permanent rest then

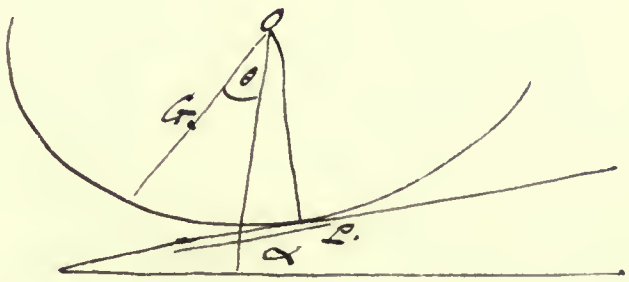
when $\phi = 0$ $\frac{d\phi}{dt} = 0$.

$$\therefore \omega^2 = 2g \left\{ -\sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha} + a \right\}$$

$$\frac{a^2+b^2}{4} + \frac{(a^2-b^2)\sin^2\alpha\cos^2\alpha}{a^2\cos^2\alpha + b^2\sin^2\alpha}.$$

A cylindrical body oscillates in a plane, inclined at a very small $\angle \alpha$ to the horizon, the friction being such as to prevent all sliding. Its radius $= a$, and the distance of the cent. of grav. from the axis being $= c$. Shew that the time of oscillation is very little different from that on a horizontal plane, but that the extent of oscillation is $\pi \frac{2a\alpha}{c}$

O the cent. of the cylinder. G. cent of grav
$\alpha$ = incln. of the plane.
$\theta$ = incln. of GO to the vertical at time $t$.




$$\therefore M\left(k^2 + OG^2\right)\frac{d^2\theta}{t} = -Mg\left(GO\sin\theta + CO\sin\alpha\right)$$

$$\frac{d^2\theta}{t} + \frac{g.GO}{k^2+AG^2}\theta = -g.\frac{CO\sin\alpha}{k^2+AG^2.}$$

$$\theta = A\cos nt + B - g.\frac{CO\sin\alpha}{k^2+AG^2}\cdot\frac{k^2+AG^2}{g.GO.}$$

$$= A\cos nt + B - g.\frac{CO.\sin\alpha}{g.GO.}$$

Let $\theta_1$, $\theta_2$ be the max and min values of $\theta$.

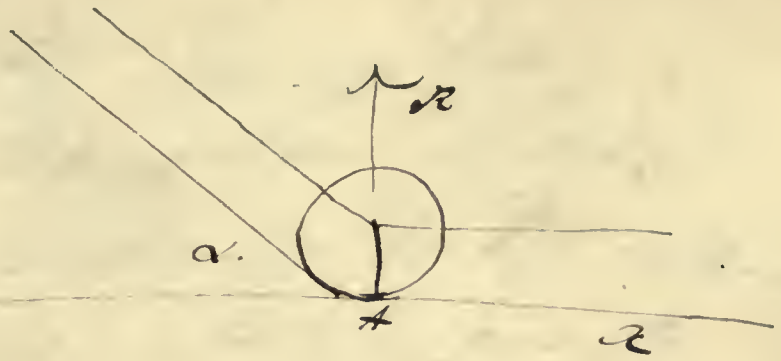
$$\therefore \theta_1 = A - \frac{CO.\ \sin\alpha}{CO.}$$

$$-\theta_2 = -A - \frac{CO\ \sin\alpha}{CO.}$$

$$\therefore \theta_1 - \theta_2 = -\frac{2\,CO.\ \sin\alpha}{CO}$$

$$= -\frac{2a\alpha}{c}.$$

A uniform homogeneous cylinder rolls down a perfectly rough inclined plane which terminates in a perfectly rough horizontal plane. find the velocity of the cylinder along the horizontal plane.




Let $u$ = velocity of cylinder before impact
$= u\cos\alpha$ // to $Ox$. $a$ = rad of cylinder
$u\sin\alpha$ ⊥ to $Ox$.
$v$ = velocity of cylinder after impact.

$\therefore m'k^2\left\{\frac{u}{a} - \frac{v}{a}\right\} = Ra.$ (1)

$m(u\cos\alpha - v) = -R = -\frac{m'k^2}{a^2}.(u-v) = -\frac{m'}{2}.(u-v)$

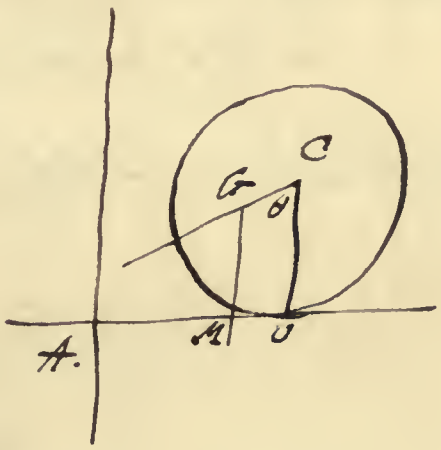
$\therefore u(2\cos\alpha - 1) = v$ $u\{2\cos\alpha + 1\} = 3v.$

$\therefore v = \frac{1}{3}u(2\cos\alpha + 1).$

$\therefore R = m(v - u\cos\alpha) = \frac{1}{3}mu(1 - \cos\alpha)$

also $R = \mu mu\sin\alpha$. since the velocity
1 in the plane often impact is zero.

An angular velocity having been impelled upon a heterogeneous sphere about an axis perpendicular to the vertical plane which contains its C of grav. G and its geometrical cent. C. and passing thro G. It is then placed upon a smooth horizontal plane: to det. the magnitude of the impelled angular velocity, that G may rise into a point in the vertical sect and there rest.




Let AH = x. HG = y
be coord$^{s}$ of G the centre
of grav. R = pressure on the
plane at D.

$$\therefore m k^2 \frac{d^2\theta}{t^2} = -Rc \sin\theta.$$

$$m \frac{d^2 y}{t^2} = R - mg.$$

$$\therefore k^2 \frac{d^2\theta}{t^2} = -c \sin\theta \left( \frac{d^2 y}{t^2} + g \right).$$

But $y = a - c\cos\theta$

$$\frac{dy}{t} = c \sin\theta \cdot \frac{d\theta}{t}$$

$$\frac{d^2 y}{t^2} = c\cos\theta \cdot \left(\frac{d\theta}{t}\right)^2 + c \sin\theta \cdot \frac{d^2\theta}{t^2}.$$



$$\therefore \left(k^2 + c^2 \sin^2\theta\right) \frac{d^2\theta}{dt^2} + c^2 \sin\theta\cos\theta \left(\frac{d\theta}{dt}\right)^2 = -cg\sin\theta.$$

$$k^2 \frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2} + 2c^2\sin^2\theta \frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2} + 2c^2\sin\theta\cos\theta\cdot\frac{d\theta}{dt}\left(\frac{d\theta}{dt}\right)^2 = -2cg\sin\theta\cdot\frac{d\theta}{dt}.$$

$$\int \sin^2\theta \frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2} = \sin^2\theta.$$

$$\int \sin^2\theta\cdot\frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2} = \sin^2\theta\cdot\frac{d\theta}{dt}\left(\frac{d\theta}{dt}\right) - \int 2\sin\theta\cos\theta\left(\frac{d\theta}{dt}\right)^2\cdot\left(\frac{d\theta}{dt}\right) - \int \sin^2\theta\cdot\frac{d^2\theta}{dt^2}\cdot\left(\frac{d\theta}{dt}\right).$$

$$= \frac{1}{2}\sin^2\theta\cdot\left(\frac{d\theta}{dt}\right)^2 - \int \sin\theta\cos\theta\cdot\left(\frac{d\theta}{dt}\right)^3.$$

$$\therefore \int \sin^2\theta\cdot\frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2} + \int \sin\theta\cos\theta\cdot\left(\frac{d\theta}{dt}\right)^3 = \frac{1}{2}\sin^2\theta\left(\frac{d\theta}{dt}\right)^2.$$

$$\therefore \left(k^2 + c^2\sin^2\theta\right)\left(\frac{d\theta}{dt}\right)^2 = 2cg\cos\theta + C.$$

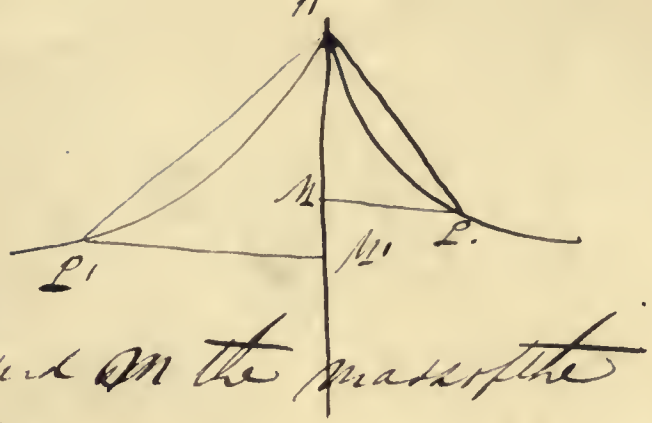
when $\sin\theta = \alpha$. $\frac{d\theta}{dt} = \omega$.

$$\therefore \left(k^2 + c^2\sin^2\alpha\right)\omega^2 = 2cg\cos\alpha + C.$$

$$\therefore k^2\left(\left(\frac{d\theta}{dt}\right)^2 - \omega^2\right) + c^2\left(\sin^2\theta\cdot\left(\frac{d\theta}{dt}\right)^2 - \sin^2\alpha\cdot\omega^2\right) = 2cg\cdot(\cos\theta - \cos\alpha).$$

when $\sin\theta = \pi$. $\frac{d\theta}{dt} = \alpha$.

$$\therefore k^2\cdot\omega^2 + c^2\sin^2\alpha\,\omega^2 = 2cg(1 + \cos\alpha)$$




Exchange of $m'$ = the mass of the body whose co-ords are $AM'$ $M'P'$; and $m$ the mass of the body whose co-ord$^{s}$ are. $AM$. $MP$.

$$m\left(\frac{ds}{t}\right)^2 + m'\left(\frac{ds'}{t}\right)^2 = 2g\left(my - m'y'\right)$$

$AM = y$. $AM' = y'$

or if $v$. $v'$ be the velocities.

$$mv^2 + m'v'^2 = 2g.\left(my - m'y'\right)$$

But $\left(\frac{v}{v'}\right)^2 = \left(\frac{ds}{ds'}\right)^2$.

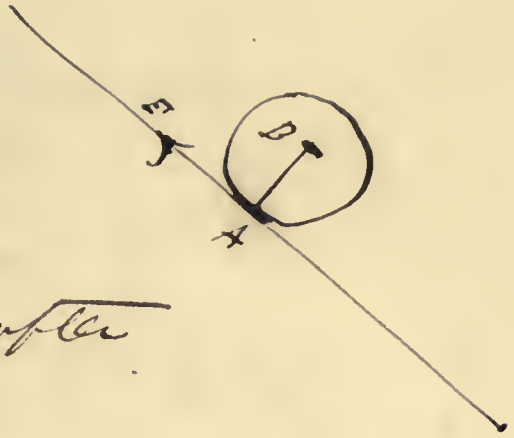
$$\therefore v^2 = v'^2 \frac{ds^2}{ds'^2}.$$

$$\therefore \left(m' + m.\frac{ds^2}{ds'^2}\right)v'^2 = 2g\left(my - m'y'\right).$$

$$v'^2 = 2g\left(my - m'y'\right).\frac{ds'^2}{m'\left(ds'^2 + m\,ds^2\right)}.$$

$$v^2 = 2g\left(my - m'y'\right)\frac{ds^2}{m'ds'^2 + m\,ds^2}.$$






Let $u$ be the velocity
the instant before sliding
$u'$ = velocity the instant after
sliding.

$$\therefore m(u - u') = +E.$$

Also. if $w$ = Ln velocity communicated.

$$mk^2 w = Fa = mk^2 \frac{u'}{a}.$$

$$\therefore E = \frac{mk^2}{a^2} u' = \frac{m}{2} u'$$

$$\therefore mu - mu' = +\frac{m}{2} u'$$

$$\therefore u = \frac{3}{2} u' \quad \therefore u' = \frac{2}{3} u$$

$$E = \frac{m}{2} u' = \frac{m}{2} \cdot \frac{2}{3} u = \frac{m}{3} u.$$

Homogeneous cylinder slides without rolling down an inclined plane which is for a certain space quite smooth. and after acquiring a given velocity is suddenly caused by the roughness of the surface to roll without sliding M. the rel. of the axis of the cylinder the instant

Let the axis major of the ⊙ be the axis of x.

Then the eqns of motion are

$$\frac{d^2x}{t} = 0 \quad \& \quad \frac{d^2y}{t} = -E. \quad (2)$$

Now $y^2 = \frac{b^2}{a^2}\sqrt{a^2-x^2}$ $\therefore \frac{dy}{x} = -\frac{b^2}{a^2}\frac{x}{\sqrt{a^2-x^2}}$

from (1) $\frac{dx}{t} = \beta$.

$$\text{also } \frac{dy}{t} = \frac{dy}{x}\frac{dx}{t} = -\frac{b^2\beta}{a^2}\cdot\frac{x}{\sqrt{a^2-x^2}}.$$

$$\frac{d^2y}{t} = \frac{dy}{t}\left\{-\frac{b^2\beta}{a^2}\frac{x}{\sqrt{a^2-x^2}}\right\}$$

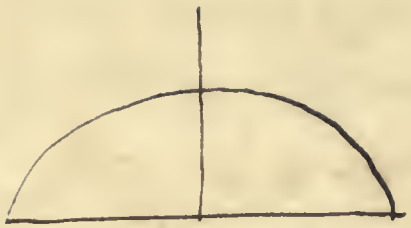
$$= d_x\left\{-\frac{b^2\beta}{a^2}\cdot\frac{x}{\sqrt{a^2-x^2}}\right\}\frac{dx}{t}.$$

$$= -\frac{b^2\beta^2}{a^2}\cdot d_x\frac{x}{\sqrt{a^2-x^2}}$$

$$= -\frac{b^2\beta^2}{a^2}\cdot\frac{a^2}{(a^2-x^2)^{\frac{3}{2}}}$$

$$= -\frac{b^4\beta^2}{a^2y^3}. \quad = E.$$

A particle describes an ⊙ under the action of a force ⊥ to its major axis. find the law of force.



Let O be base of the cycloid better as origin.

Then for the motion we have.

$$\frac{d^2x}{t^2} = 0. \quad // \quad \frac{d^2y}{t^2} = F.$$

Now $x = a(1-\cos\theta) \qquad y = a(\theta + \sin\theta)$.

$$\frac{dx}{t} = a\sin\theta \qquad \frac{dy}{t} = a(1+\cos\theta)$$

$$\frac{\frac{dy}{t}}{\frac{dx}{t}} = \frac{dy}{x} = \frac{1+\cos\theta}{\sin\theta}.$$

Now $\frac{dy}{t} = \frac{dy}{x}\frac{dx}{t} = \beta.\ \frac{1+\cos\theta}{\sin\theta}.$

$$\frac{d^2y}{t^2} = \frac{d}{t}\left\{\beta.\ \frac{1+\cos\theta}{\sin\theta}\right\} = \frac{d}{\theta}\left\{\beta\ \frac{1+\cos\theta}{\sin\theta}\right\}.\frac{d\theta}{t}.$$

Now $\frac{dx}{t} = \beta = \frac{dx}{\theta}\frac{d\theta}{t} = a\sin\theta.\frac{d\theta}{t}.$

$$\therefore \frac{d\theta}{t} = \frac{\beta}{a\sin\theta}.$$

$$\therefore \frac{d^2y}{t^2} = \frac{\beta^2}{a\sin^3\theta}\left\{-\sin^2\theta + (1+\cos\theta)\cos\theta\right\}$$

$$= \frac{2\beta^2\cos^2\theta}{a\sin^3\theta}.$$

$$\frac{1}{F} = \frac{a}{2\beta^2}\left(\sin^2\theta\sin\theta\right)$$

$$= \frac{a}{2\beta^2}\left\{2\sin^3\theta - \sin 2\theta\right\}.$$

A particle describes an arc of a cycloid under the action of a force at right angles to the major axis. Find the force at any point of the orbit in terms of its distance from its base.

II. If the angular velocity be variable then $\frac{dw}{t}$ is the $\perp^{r}$ accelerat. force; and the linear acc$^{n}$. force in direct$^{n}$. of motion is $c.\frac{dw}{t}$. The acc$^{n}$. force $\perp$ to the direct$^{n}$. of motn is as before $cw^{2}$. $\therefore$ the resultant of these two is $c\sqrt{w^{4}+(\frac{dw}{t})^{2}}$. applied at G. and moment of this about A is

$c\,GM.\sqrt{w^{4}+(\frac{dw}{t})^{2}}$. which must be $= g\,AM$.

$$\therefore \frac{AM}{GM} = \frac{c}{g}\cdot\sqrt{w^{4}+(\frac{dw}{t})^{2}}.$$

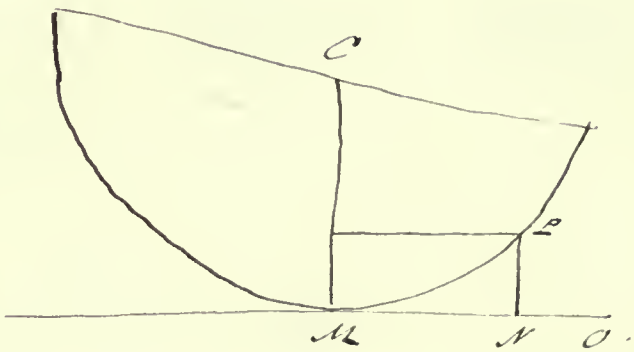
$$\frac{b+ck}{x^3+a^3} = \frac{A}{x+a} \cdot \frac{L}{a}.$$

$$b+ck = A.(x^2-ax+a^2) + L.(x+a).$$

$$b+x = -a.$$

$$b-ac = A.$$

A thin hemispherical shell whose rad = a rests on a smooth horizontal plane, a particle of the same mass as the shell is placed at the lowest point of its internal surface. Show that if the shell be projected with a horizontal velocity = $2\sqrt{ga}$. the particle will ascend just as high as the vertex of the shell and then descend.




$OM = z$. $ON = x$. $NP = y$

Coord$^{s}$ of the particle

By Conservation of the motion of the C.G. of gravity.

$$M\frac{dz}{dt} + m\frac{dx}{dt} = Mu \quad (1) \quad = M \cdot 2\sqrt{ga} \quad (1)$$

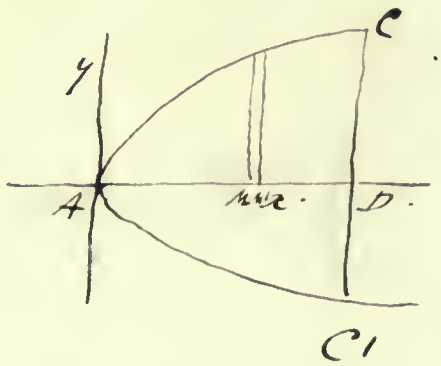
Vis Viva

$$M\left(\frac{dz}{dt}\right)^2 + m\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\} = m(C - 2gy).$$

$$M\,4ga = m\,C.$$

$$\therefore \left(\frac{dz}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4g(2a - y) \quad (2).$$

To find the rad. of gyr. of a port. of a P. bounded by a double ordinate to the axis about a ⊥ line thro' the vertex.




A the vertex. Ay. & Ax the axes of y & x.
the rad of gyr. about the perpendicular axis = rad of gy. about Ay. + rad of gyr. about Ax.
To find $k^2$ for the axis of y. mehane

$$mk^2 = \rho \int_a^x x^2 y\,dx = \rho \int_a^x 2\sqrt{m}\, x^{\frac{5}{2}} = \rho \cdot \frac{2\sqrt{m}\, x^{\frac{7}{2}}}{7} = \frac{2x^2}{7} \rho x y;$$

$$m = \frac{2}{3}\rho x y \quad \therefore k^2 = \frac{3}{7} x^2.$$

To find $k'$ for the axis of y.

$$mk'^2 = \rho \int_y^{y'} y^2 (x' - x) = \rho \int_y^{y'} \left\{ y^2 x' - \frac{y^4}{4a} \right\}.$$

$$= \rho \left\{ \frac{y'^3}{3} x' - \frac{y'^5}{20a} \right\} = \rho x' y' \left\{ \frac{y'^2}{3} - \frac{y'^2}{5} \right\}.$$

$$= \rho x' y' \cdot \frac{2}{15} y'^2.$$

$$m = \frac{2}{3} \rho x' y' \quad \therefore k'^2 = \frac{1}{5} y'^2.$$

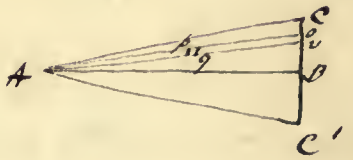
$\therefore$ the required rad of gyration $= k^2 + k'^2 =$

$$= \frac{3}{7} x'^2 + \frac{1}{5} y'^2.$$



To find the radius of gyration of a regular polygon, about an axis + laid thro' its centre.

Let $CC' = c$. and the polygon
be of $n$ sides $\therefore \angle CAD = \frac{\pi}{n}$.
The moment of inertia of an element $pq$.
about the proposed axis is $m.r^2 \delta\theta.Or$.




$\therefore \int_r m r^3 \delta\theta = \frac{m}{4} r^4 . \delta\theta + C$

take the integral from $r = 0$ & $r = AB \sec\theta$.

$\therefore \frac{m}{4} . AB^4 \sec^4\theta \, \delta\theta$ = moment of inertia of the slice $AOv$.

$\therefore \frac{m}{4} . AB^4 . \int \tan\theta \{1 + \tan^2\theta\} = \frac{m}{4} AB^4 . \{\tan\theta + \frac{1}{3}\tan^3\theta\} + C$

take the integral from $\theta = \frac{\pi}{n}$ to $\theta = -\frac{\pi}{n}$. and we have for the moment of inertia of the whole $\triangle ACC'$

$$\frac{m}{2} . AB^4 . (\tan\frac{\pi}{n} + \frac{1}{3}\tan^3\frac{\pi}{n}) = \frac{m}{2} . \tan^3\frac{\pi}{n} . AB^4 \{\frac{1}{\tan^2\frac{\pi}{n}} + \frac{1}{3}\}.$$

$\therefore$ moment of inertia of the whole polygon =

$$\frac{nm}{2} . AB^4 \tan^3\frac{\pi}{n} \{\frac{3 + \tan^2\frac{\pi}{n}}{3\tan^2\frac{\pi}{n}}\}.$$

also mass of the polygon $= n(AB.BC)m$.

$$\therefore r^2 = \frac{AB^3 \tan^3\frac{\pi}{n}}{6.BC} \{\frac{3\cos^2\frac{\pi}{n} + \sin^2\frac{\pi}{n}}{\sin^2\frac{\pi}{n}}\}$$

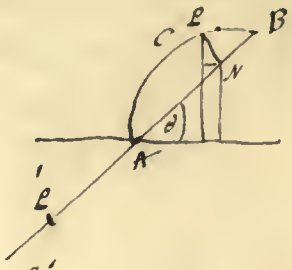
$$= \frac{BC^2}{3} . \{\frac{2\cos^2\frac{\pi}{n} + 1}{2\sin^2\frac{\pi}{n}}\} \quad \text{for } BC = AB\tan\frac{\pi}{n}.$$

$$= \frac{CC'^2}{12} \{\frac{2 + \cos\frac{2\pi}{n}}{1 - \cos\frac{2\pi}{n}}\} = \frac{c^2}{12} \{\frac{2 + \cos\frac{2\pi}{n}}{1 - \cos\frac{2\pi}{n}}\}$$

$$\frac{\frac{1}{h}}{1+\frac{x^2}{h^2}} \cdot \frac{h}{h^2+x^2}$$

$$\frac{\frac{1}{h}}{\sqrt{1-\frac{x^2}{h^2}}} \cdot \frac{1}{\sqrt{h^2-x^2}}$$

BB' is an inflexible rod, to one end of which is attached a semi-elliptical table ACB. BB' is moveable about B in a vertical plane, which also coincides with the plane of the table. Two particles a. b. are placed at C and are capable of moving along BCA. the other at B' capable of moving along AB'. determine the motion of each particle.




Let P be the place of a at the time t. AN = x. NP = y.

P' ——— b ——— AP' = r.

by the principle of Vis Viva.

$$a\left(x^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right) + br^2\left(\frac{d\theta}{dt}\right)^2 = C'.$$

by the principle of conservation of areas.

$$\left(a.x^2 + br^2\right)\left(\frac{d\theta}{dt}\right) = C''.$$

now $\left(\frac{dx}{dt}\right)^2 = \left(\frac{dx}{d\theta}\right)^2\left(\frac{d\theta}{dt}\right)^2$. ; $\left(\frac{dy}{dt}\right)^2 = \left(\frac{dy}{d\theta}\right)^2\left(\frac{d\theta}{dt}\right)^2$.

$$\therefore a\left(x^2 + \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right) + br^2 = \frac{C'}{\left(\frac{d\theta}{dt}\right)^2}$$

$$= \frac{C'}{C''^2}.\left(ax^2 + br^2\right)^2$$

$$a\left\{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right\} = \left\{\frac{C'}{C''^2}\left(ax^2 + br^2\right) - 1\right\}\left(ax^2 + br^2\right).$$

$$\text{or } \frac{a'^2 - e^2x^2}{a'^2 - x^2}.\left(\frac{dx}{d\theta}\right)^2 = \left\{\frac{C'}{C''^2}\left(ax^2 + br^2\right) - 1\right\}\left\{x^2 + \frac{b}{a}r^2\right\} \quad (A).$$

Also $h^2 \frac{d^2\theta}{dt^2}.(a+b) = a\{x\cos\theta - y\sin\theta\}g. + b\cos\theta g$

$$= a\left(x\cos\theta - \frac{b}{a}\sqrt{a^2 - x^2}.\sin\theta\right)g. + b\cos\theta g.$$

$$\therefore (a+b)h^2\left(\frac{d\theta}{dt}\right)^2 = 2a.\left\{x\sin\theta + \frac{b}{a}\sqrt{a^2 - x^2}.\cos\theta\right\}g + b\sin\theta g. \quad \therefore d\theta = f(x.\theta) = \frac{C''}{ax^2 + br^2}$$

substitute in # and reduce the difft. eqn. between $\frac{dx}{d\theta}$ and $x$.

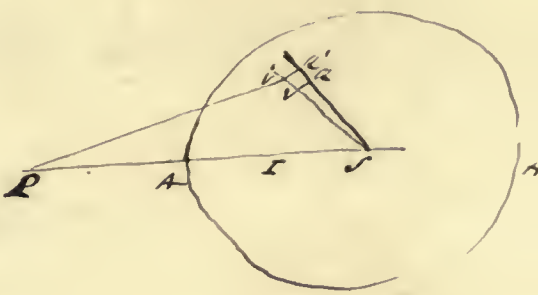
$$\frac{a'^2 - e^2x^2}{a'^2 - x^2}.\left(\frac{dx}{d\theta}\right)^2 = \frac{C'}{\quad}\left(\frac{1}{f(x.\theta)} - \frac{1}{C''}\right)\left\{\frac{1}{f(x.\theta)}\right. \cdot \text{ or } \left(\frac{dx}{d\theta}\right)^2 = \frac{C''.f(x.\theta)}{f(x.\theta)^2}.\frac{C'}{C''}.\frac{a'^2 - x^2}{a'^2 - e^2x^2}.$$

in the cluster. SA is the radius of a sphere. Each of whose particles has an attractive force varying as $\frac{1}{u^n}$. Having assumed in SA produced any point L and having taken SL : SA :: SA : SL' find the ratio of the attraction which the whole sphere exerts on equal corpuscles placed at A and L.

A is a pt. of the sph. made by a plane passing thro' S, $SC = b$. $SA = a$ $\therefore SL = \frac{a^2}{b}$

$SV = SQ = r$. $SV' = SQ' = r + \delta r$. $\angle CSV = \theta$.

$\angle CSQ = \theta + \delta\theta$




Then the force of attract: at a distance $u = \mu u^n$

whose chord $u^2 = b^2 + r^2 - 2br\cos\theta$.

The mass of the element $= \rho r\delta r.\delta\theta$.

mass of annulus generated by the revolut: of this element about the axis $= 2\rho\pi.r^2\delta r.\delta\theta.\sin\theta = 2\pi\rho.r^2\delta r.\delta(\cos\theta) = \frac{2\pi\rho}{b}.r\delta r.u.\delta u$.

The attract: of this annulus in direct: $LS = \frac{2\pi\rho}{b}.r.\delta r.u\,\delta u\left\{\frac{u^2+b^2-r^2}{2bu}\right\}$.

$\therefore$ attract: of the sphere in the same direct: on the particle $= \left\{\frac{\pi\rho}{b^2}\int r\int u^r\left\{u^{n+2} + (b^2-r^2)u^n\right\}\right\}$.

$$= -\frac{\pi\rho}{b^2}\cdot\int r\cdot r\left\{\frac{(b+r)^{n+3}-(b-r)^{n+3}}{n+3} + (b^2-r^2)\frac{(b+r)^{n+1}-(b-r)^{n+1}}{n+1}\right\}$$

$$= -\frac{\pi\rho}{b^2}\left\{\int r\cdot\frac{(b+r)^{n+4} - b(b+r)^{n+3} + (b-r)^{n+4} - (b-r)^{n+3}}{n+3}\right.$$

$$\left. + \int r\left\{b-r.\frac{(b+r)^{n+2} - (b+r)(b-r)^{n+2}}{n+1}\right\}\right\}.$$

$$= -\frac{\pi\rho}{b^2}\int r\left\{\frac{(b+r)^{n+4} - b.(b+r)^{n+3} + (b-r)^{n+4} - (b-r)^{n+3}}{n+3} - \frac{1}{n+1}\left((b+r)^{n+4} - 3b(b+r)^{n+3} + 2b^2(b+r)^{n+2}\right.\right.$$
$$\left.\left.(b-r)^{n+4} - 3b(b-r)^{n+3} + 2b^2(b-r)^{n+2}\right)\right.$$

$$= -\frac{\pi\rho}{b^2}\left\{\frac{1}{n+3}\left\{\frac{(b+a)^{n+5}}{n+5} - 2.\frac{(b+a)^{n+4}}{n+4} - \frac{(b-a)^{n+5}}{n+5} + \frac{2(b-a)^{n+4}}{n+4}\right\} - \frac{1}{n+1}\cdot\left\{\frac{(b+a)^{n+5} - (b-a)^{n+5}}{n+5}\right.\right.$$
$$\left.\left. - \frac{3b}{n+4}\left\{(b+a)^{n+4} - (b-a)^{n+4}\right\} + \frac{2b^2}{n+3}\left\{(b+a)^{n+3} - (b-a)^{n+3}\right\}\right.\right.$$

$$\frac{\pi\rho}{b^2}\left\{\frac{(b+a)^{n+5} - (b-a)^{n+5}}{(n+5)(n+3)(n+1)} - \frac{(3n+8)b\left\{(b+a)^{n+4} - (b-a)^{n+4}\right\}}{(n+5)(n+3)(n+1)}\right.$$
$$\left. + \frac{2b^2}{(n+1)(n+3)}\cdot(b+a)^{n+3} - (b-a)^{n+3}\right\}.$$

$$\frac{\pi\rho}{(n+5)b^2(n+3)(n+1)}\left\{2\left\{(b+a)^{n+5} - (b-a)^{n+5}\right\} - (3n+8)\left\{b.(b+a)^{n+4} - (b-a)^{n+4}\right\} + 2(n+5)b^2\left\{(b+a)^{n+3} - (b-a)^{n+3}\right\}\right.$$

for the attract. of the particle of I. mechanics also.

$$A = \frac{\pi\rho}{b'^2} \cdot \int r \int a^n \left( a^{n+2} + \left(b'^2 - r^2\right) a^n \right)$$

$$= \frac{\pi\rho}{b'^2} \cdot \int r \cdot r \left\{ \frac{\left(r-b'\right)^{n+3} - \left(r+b'\right)^{n+3}}{n+3} + \frac{\left(b'^2 - r^2\right)}{n+1} \cdot \left(r-b'\right)^{n+1} - \left(r+b'\right)^{n+1} \right.$$

where $b' = \frac{a^2}{b}$.

$$\frac{\pi\rho \cdot b^2}{a^4} \cdot \int r \left\{ \frac{\left(r - \frac{a^2}{b}\right)^{n+4} + \frac{a^2}{b}\left(r - \frac{a^2}{b}\right)^{n+3} - \left(r + \frac{a^2}{b}\right)^{n+4} + \frac{a^2}{b}\left(r + \frac{a^2}{b}\right)^{n+3}}{n+3} \right.$$

$$\left( \frac{\pi\rho b^2}{a^4} \int r \left\{ -\left(\frac{a^2}{b} + r\right)\left(r - \frac{a^2}{b}\right)^{n+2} - \left(\frac{a^2}{b} - r\right)\left(r + \frac{a^2}{b}\right)^{n+2} \right\} \right.$$

$$\frac{\pi\rho b^2}{a^4} \left\{ \frac{\left(\frac{2a-a^2}{b}\right)^{n+5}}{n+5} + \frac{a^2}{b} \cdot \frac{\left(\frac{2a-a^2}{b}\right)^{n+4} - \left(\frac{2a+a^2}{b}\right)^{n+4}}{n+4} - \left(\frac{ab+a^2}{b}\right)^{n+5} \frac{1}{n+4} \right\} \cdot \frac{1}{n+3}$$

$$+ \frac{\pi\rho b^4}{(n+1)a^4} \left\{ \frac{\left(\frac{ab-a^2}{b}\right)^{n+5} - \left(\frac{ab+a^2}{b}\right)^{n+5}}{n+5} + \frac{3a^2}{b} \cdot \frac{(ab-a^2)^{n+4} - (ab+a^2)^{n+4}}{n+4} + \frac{2a^4}{b^2} \cdot \frac{(ab-a^2)^{n+3} - (ab+a^2)^{n+3}}{n+3} \right.$$

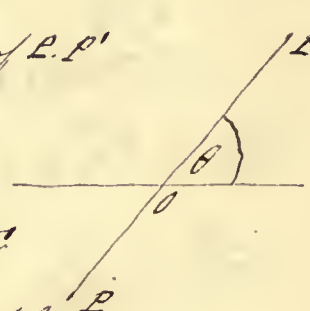
$$\frac{\pi\rho b^4}{(n+5)a^2(n+3)(n+1)} \left\{ 2\left(\frac{2a+a^2}{b}\right)^{n+5} - \left(\frac{2a-a^2}{b}\right)^{n+5} - (3n+8)\left(\frac{2a+a^2}{b}\right)^{n+4} - \left(\frac{2a-a^2}{b}\right)^{n+4} \right.$$

$$\left. \left. + 2(n+5)\left\{\left(\frac{2a+a^2}{b}\right)^{n+3} - \left(\frac{2a-a^2}{b}\right)^{n+3}\right\} \frac{b^4}{a^2} \right\} \right\} \cdot \quad B.$$

$\therefore$ the attractions are as $A : B$.

Two particles $P P'$ are connected together by a rigid rod without inertia which passes through a small ring at $O$; the rod rests upon a horizontal plane. Suppose any impulse communicated to the particles find the paths they will describe.

$\because OP = r, \quad OP' = l - r \quad \therefore PP' = l.$ $m, m'$ masses of $P, P'$




Then by principle of vis viva

$$m\left\{\left(\frac{dr}{t}\right)^2 + r^2\left(\frac{d\theta}{t}\right)^2\right\} + m'\left\{\left(\frac{d}{t}.l-r\right)^2 + (l-r)^2\left(\frac{d\theta}{t}\right)^2\right\} = C$$

when $t = 0$ $\frac{dr}{t} = \beta$, $r = a$, $l - r = a'$. $\frac{d\theta}{t} = \omega$.

$$\therefore (m + m')\beta^2 + (ma^2 + m'a'^2)\omega^2 = C$$

$$\therefore (m + m')\left(\frac{dr}{t}\right)^2 + \left\{mr^2 + m'(l-r)^2\right\}\left(\frac{d\theta}{t}\right)^2 = (m+m')\beta^2 + (ma^2 + m'a'^2)\omega^2 \ (C)$$

by principle of Conservation of Areas.

$$\left\{mr^2 + m'(l-r)^2\right\}\frac{d\theta}{t} = C' = (ma^2 + m'a'^2)\omega.$$

Now since $\frac{dr}{t} = \frac{dr}{d\theta}\cdot\frac{d\theta}{t}$ we have

$$\left(\frac{dr}{t}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 \cdot \frac{(ma^2 + m'a'^2)^2\omega^2}{\left\{mr^2 + m'(l-r)^2\right\}^2}.$$

$$\therefore (m+m')\left(\frac{dr}{d\theta}\right)^2 + mr^2 + m'(l-r)^2 = \frac{C\left\{mr^2 + m'(l-r)^2\right\}^2}{(ma^2 + m'a'^2)^2\omega^2}.$$

$$\text{or } (m+m')\left(\frac{dr}{d\theta}\right)^2 = \left\{A\left[mr^2 + m'(l-r)^2\right] - 1\right\}\left\{mr^2 + m'(l-r)^2\right\}$$

where $A = \frac{C}{(ma^2 + m'a'^2)^2\omega^2} = \frac{(m+m')\beta^2 + (ma^2 + m'a'^2)\omega^2}{(ma^2 + m'a'^2)^2\omega^2}$

$E = \frac{\mu}{p^2}$ a body is projected from an apse with a velocity which is $\sqrt{2}$ times the velocity in a circle at the same distance as $\sqrt{2} : \sqrt{3}$. find the orbit described and the time before the body falls to the centre.

$$h^2 u^2 \left(\frac{d^2u}{d\theta^2} + u\right) = \frac{\mu}{p^2} u^4.$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} u^2$$

$$= \frac{3a}{2} \cdot u^2.$$

$$\therefore \left(\frac{du}{d\theta}\right)^2 + u^2 = a u^3 + C.$$

when $\frac{du}{d\theta} = 0$ $u = \frac{1}{a}$

$$\therefore \frac{1}{a^2} = \frac{1}{a^2} + C \therefore C = 0.$$

$$\left(\frac{du}{d\theta}\right)^2 = u^2 \cdot (au - 1) \quad \text{or} \quad d\theta = \frac{1}{u\sqrt{au-1}}. \quad \therefore \theta = \int \frac{du}{u\sqrt{au-1}}$$

Let $\sqrt{au-1} = z \therefore \frac{du}{\sqrt{u-1}} = 2z\,dz.$

$u = z^2 + 1$

$$\therefore \frac{du}{u\sqrt{u-1}} = \frac{2z\,dz}{z^2+1}.$$

$$\therefore \theta = 2\int \frac{1}{z^2+1} = 2\tan^{-1} z. \quad C = 0.$$

$$\tan\frac{\theta}{2} = z = \sqrt{au-1}.$$

$$\sec^2\frac{\theta}{2} = \frac{a}{r}.$$

$$\therefore r = a\cos^2\frac{\theta}{2}.$$

$$= \frac{a}{2}(\cos\theta + 1)$$

when $\cos\theta = 1$ or $\theta = 0$.

$\{$ In a $\odot$ $v = \sqrt{E \cdot R} = \sqrt{\frac{\mu}{p}}$ in this case

$$\therefore v' = \sqrt{\frac{2\mu}{3r^3}}.$$

at the apse $\cos\theta = a$.

$$\therefore h^2 = v'^2 a^2 = \frac{2\mu}{3a}.$$

$$\frac{1}{2}\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right) = v^2 = 2\int \frac{\mu}{r^4}$$

$$\frac{a^2}{4}\{1 - \cos^2\theta + \cos^2\theta - 2\cos\theta + 1\}\left(\frac{d\theta}{dt}\right)^2 = v^2.$$

$$\{1 - \cos\theta\}\left\{\left(\frac{d\theta}{dt}\right)^2 = v^2 + \frac{2}{3}\frac{\mu}{r^3}\right.$$

$$= v^2 + \frac{2}{3}\frac{\mu}{a^3} \cdot \frac{2\sec^2\frac{\theta}{2}}{\left(1 + \tan^2\frac{\theta}{2}\right)^2}.$$

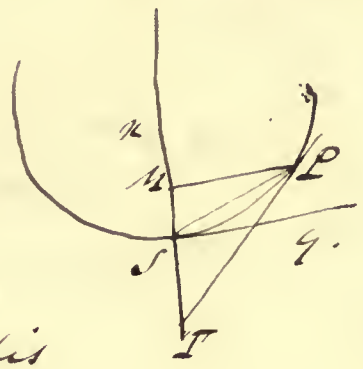
$$\left(\frac{d\theta}{dt}\right)^2 = \frac{v^2}{a^2\cos^2\frac{\theta}{2}} + \frac{2}{3}\frac{\mu}{a^3} \cdot 2\sec^2\frac{\theta}{2}\left(1 + \tan^2\frac{\theta}{2}\right)^2.$$

$$= \frac{2\sec^2\frac{\theta}{2}}{a^2}\left\{v^2 + \frac{1}{3}\frac{\mu}{a^3} \cdot 2\sec^6\frac{\theta}{2}\right\}.$$

$$\therefore at = \frac{a}{2\sec^2\frac{\theta}{2}} \cdot \frac{1}{\sqrt{v^2 + \frac{2}{3}\frac{\mu}{a^3}\sec^6\frac{\theta}{2}}}.$$

$$\therefore t = a\int_0^\theta \frac{\cos^2\frac{\theta}{2}}{\sqrt{v^2 + \frac{2}{3}\frac{\mu}{a^3}\sec^6\frac{\theta}{2}}}$$

A centre of force which is repulsive and varies as $(dist)^n$ is situated at the lowest point of a vertical O. how far will a body placed at S and acted on by gravity ascend up the curve.




P. the position of the body at the time $t$. $SP = r$. $SM = x$. $MP = y$.

The force attracting along the tangent is

$$v\,dv = -\frac{\mu}{r^n}\,dr + g\,dx$$

$$\int$$

$$\therefore \frac{1}{2}v^2 = \frac{\mu}{n-1}\cdot\frac{1}{r^{(n-1)}} + gx + C.$$

when $r=0$ $v=0$ $x=0$ $\therefore C=0$

$$\frac{1}{2}v^2 = \frac{\mu}{n-1}\cdot\frac{1}{r^{(n-1)}} + gx.$$

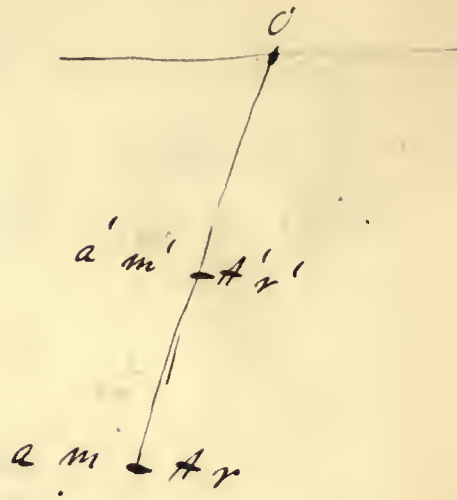
$$= \frac{\mu}{n-1}\cdot\frac{1}{(2ax)^{\frac{n-1}{2}}} + gx \quad \text{for } r^2 = x^2+y^2 = 4ax.$$

when $v=0$ the body arrives at the highest point

$$\therefore 0 = \frac{\mu}{n-1}\cdot\frac{1}{(2a)^{\frac{n-1}{2}}} + g.x.^{\frac{n+1}{2}}.$$

$$\therefore x = \sqrt[n+1]{\left\{\frac{\mu}{(n-1)g}\cdot\frac{1}{(2a)^{\frac{n-1}{2}}}\right\}^2}.$$

A pendulum consists of an indefinite thin rod OA, and a globe, of which the centre is A; but determine the point A in the line OA at which the centre of another globe must be fixed in order that the oscillations of the globes may be executed in the shortest time.




Let r' = rad of A' ∴ $\frac{2}{5}r'^2$ = rad of gyr. about diam.

$\frac{2}{5}r'^2 + a'^2$ = ——— about the axis thro O

$r$ = rad of A ∴ $\frac{2}{5}r^2 + a^2$ = ——— do.

$\therefore k^2 = a^2 + a'^2 + \frac{2}{5}(r'^2 + r^2)$.

also $k(m+m') = m'a' + ma$ ∴ $k = \frac{m'a' + ma}{m+m'}$

∴ length of isochronous pendulum $= \frac{k^2 + k^2}{k}$

$= \frac{(m+m')\left\{a^2 + a'^2 + \frac{2}{5}r^2 + r'^2\right\}}{m'a' + ma}$.

for the shortest oscillations this length must be a max.

$\therefore d_{a'}\left\{\frac{(m+m')\left\{a^2 + a'^2 + \frac{2}{5}(r'^2 + r^2)\right\}}{m'a' + ma}\right\} = 0$ gives the required length of $a'$.

A body is projected in the plane $(x,y)$ with a velocity $(v)$ in a direction making an angle $\alpha$ with the axis of $(x)$ and is acted on by a force $\mu x^n$ // to the axis of $y$. Find the $n^{th}$ path or orbit described.

$$\text{Here } \frac{d^2y}{t^2} = \mu x^n. \quad (A)$$

$$\frac{d^2x}{t^2} = 0 \quad \therefore \frac{dx}{t} = v\cos\alpha.$$

$$x = vt\cos\alpha. \quad (1)$$

Substitute in $A$ and we have.

$$\frac{d^2y}{t^2} = -\mu v^n t^n \cos^n\alpha.$$

$$\therefore \frac{dy}{t} = -\frac{\mu}{n+1} \cdot v^n \cos^n\alpha \cdot t^{n+1} + v\sin\alpha.$$

$$y = -\frac{\mu}{(n+1)(n+2)} \cdot v^n \cos^n\alpha \cdot t^{n+2} + vt\sin\alpha.$$

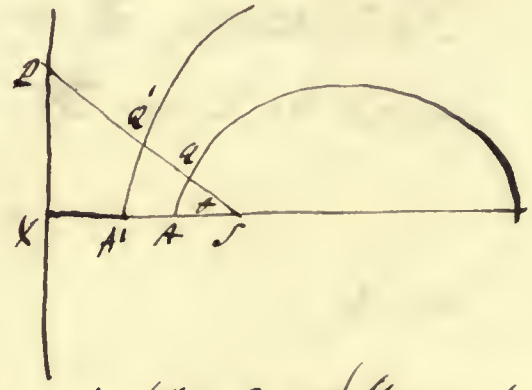
$$= vt\sin\alpha - \frac{\mu.(v\cos\alpha.t)^{n+2}}{(n+1)(n+2)v^2\cos^2\alpha}$$

but $v\cos\alpha.t = x \quad \therefore t = \frac{x}{v\cos\alpha}.$

$$\therefore y = x\tan\alpha - \frac{\mu x^{n+1}}{(n+1)(n+2)v^2\cos^2\alpha}.$$

A particle moves from any point in the directrix of an ellipse in a straight line towards a centre of force $\propto \frac{1}{(\text{dist})^2}$ in the nearer focus. find its velocity when it arrives at the curve.

$S$ the focus. $P$ the foot in the directrix




$AS = a(1-e)$. $AX = \frac{a(1-e)}{e}$ $\therefore SX = \frac{a(1-e^2)}{e}$

$SP = \frac{a(1-e^2)}{e\cos\theta}$.

For the motion along $SP$. $\frac{d^2r}{dt^2} = -\frac{\mu}{r^2}$ $\therefore \left(\frac{dr}{dt}\right)^2 = 2\left(\frac{\mu}{r} + \text{const}\right)$

at the commencement of motion $r = SP$. and $\frac{dr}{dt} = \frac{a(1-e^2)}{1+e\cos\theta}$

$\therefore$ if $V$ = vel. at $Q$ $V^2 = 2\mu\left\{\frac{1+e\cos\theta}{a(1-e^2)} - \frac{e\cos\theta}{a(1-e^2)}\right\}$.

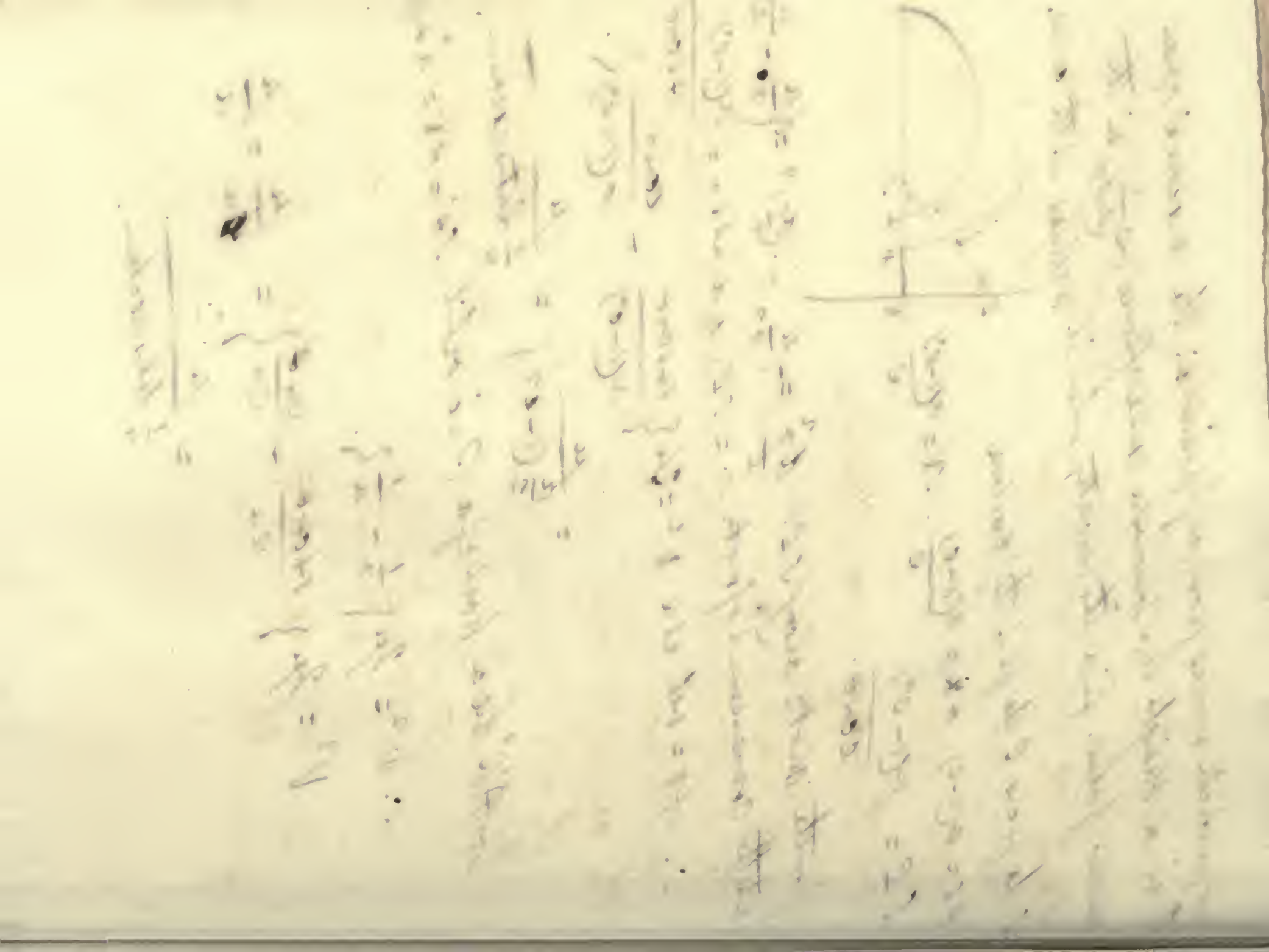
$= \frac{\mu}{\frac{a}{2}(1-e^2)} = \frac{\mu}{\frac{1}{4}\text{ latus rectum}}$.

If $A'Q'$ be a part of a circle orbit $SA' = A'X = a'$ suppose

$\therefore v^2 = 2\mu\left\{\frac{1}{r} - \frac{1}{a}\right\}$.

$V^2 = 2\mu\left\{\frac{1+\cos\theta}{2a} - \frac{\cos\theta}{2a}\right\} = \frac{\mu}{2a} = \frac{\mu}{a}$

$= \frac{\mu}{\frac{1}{4}\text{ latus rectum}}$.



At one end C of a straight tube CA is a centre of force $= \mu \cdot (\text{dist})$. The tube revolves in one plane round C with an angular velocity $= \sqrt{\frac{\mu}{2}}$. and the body descends down the tube under the actn of the force in C. Then the body's path will be a ⊙ on the initial posn of the tube.

CA the initial posn of the tube

CA' any other posn.

P the place of the particle at the time t.




$\angle ACA = \theta$. $CP = r$. $CA = a$.

Now the velocity of $P = r\sqrt{\frac{\mu}{2}}$.

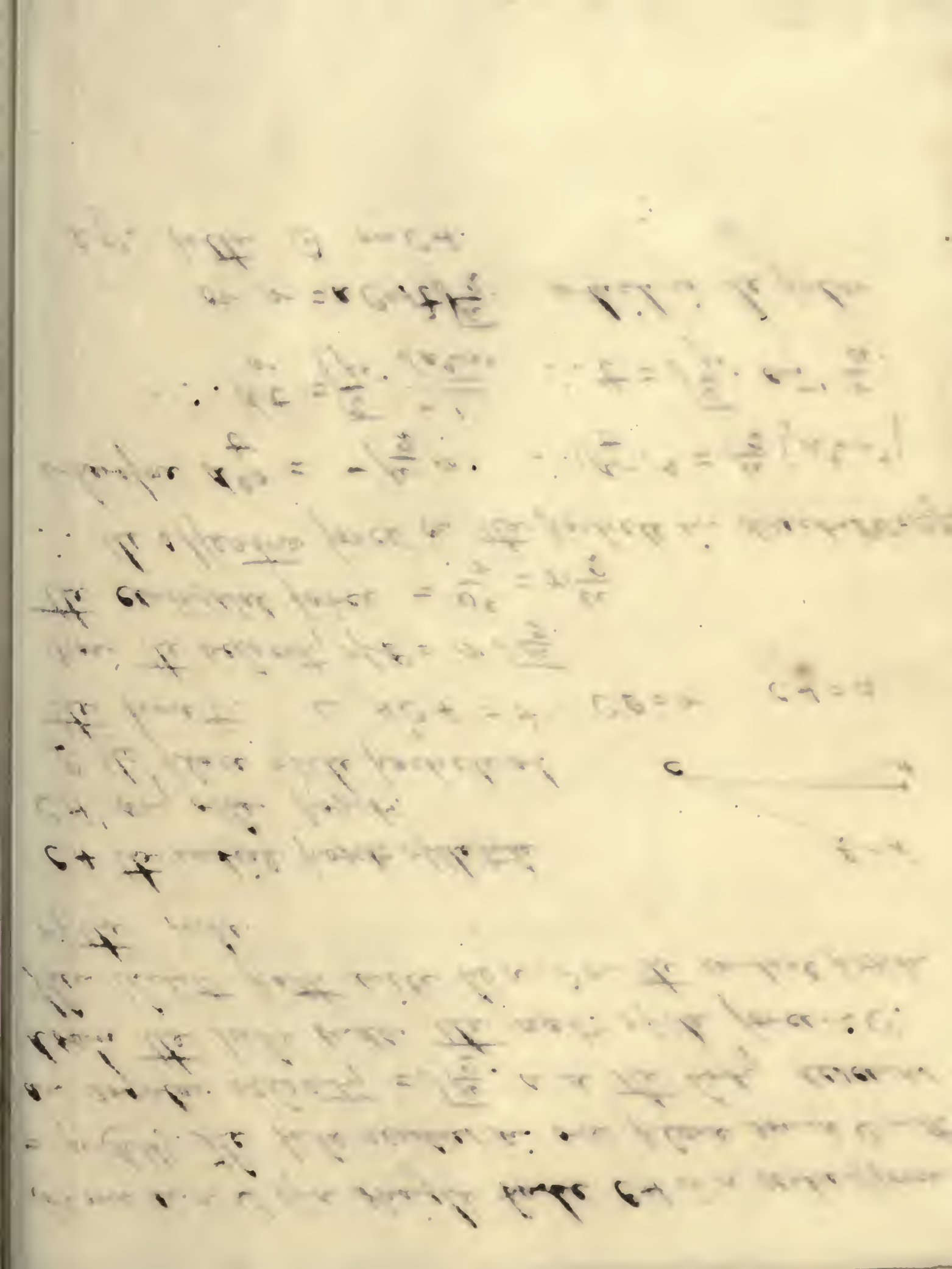
the centripetal force $= \frac{v^2}{r} = r\frac{\mu}{2}$

$\therefore$ the effective force on the particle in direction $PC = \frac{\mu r}{2}$.

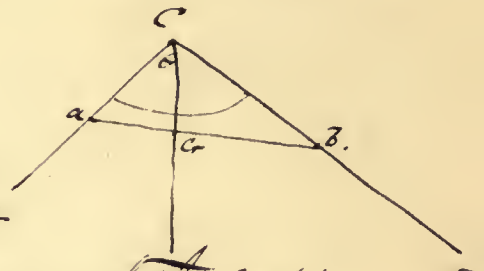
wherefore $\frac{d^2r}{t} = -\frac{\mu}{2} r$. $\therefore \left(\frac{dr}{t}\right)^2 = \frac{\mu}{2}\{a^2 - r^2\}$

$\therefore \frac{dt}{r} = \sqrt{\frac{2}{\mu}} \cdot \frac{-1}{\sqrt{a^2 - r^2}}$ $\therefore t = \sqrt{\frac{2}{\mu}} \cdot \cos^{-1}\frac{r}{a}$.

or $r = a\cos t\sqrt{\frac{\mu}{2}}$. which is the polar eqn. to the ⊙ on CA.



A heavy lever of which the arms are of length a. b. and the angle between them $\theta$, makes small oscillations in its own plane about the angular point. find the length of the isochronous simple pendulum.




Here $l = \frac{h^2+k^2}{h}$.

Let a. b. be the middle point of CA. CB. Join ab. & take $aG = \frac{b}{a+b} \cdot ab$. then G is the cent. of grav. of the rods.

Now $ab = \sqrt{\frac{a^2}{4} + \frac{b^2}{4} - \frac{ab}{2}\cos\theta}$.

$h^2 = CG^2 = (Ca)^2 + (aG)^2 - 2Ca \cdot aG \cdot \cos Ca.b$.

where $\cos Cab = \frac{\frac{a^2}{4} + (ab)^2 - \frac{b^2}{4}}{a \cdot (ab)}$.

$\therefore$ after reduction where.

$$h^2 = CG^2 = \frac{1}{4(a+b)^2} \cdot (a^4 + b^4 + 2a^2b^2\cos\theta)$$

$$\text{or } h = \frac{1}{2(a+b)} \cdot \sqrt{a^4+b^4+2a^2b^2\cos\theta}.$$

also rad of gyr. of CA about the axis of rotation $= \frac{1}{3}a^2$.

" " " CB " " " " $= \frac{1}{3}b^2$.

$$\therefore h^2 + k^2 = k_1^2 = \frac{1}{3}(a^2+b^2).$$

$$l = \frac{h^2+k^2}{h} = \frac{\frac{2}{3} \cdot (a^2+b^2)(a+b)}{\sqrt{a^4+b^4+2a^2b^2\cos\theta}}.$$

$v^2 = (1+\varepsilon)\sqrt{\frac{\mu \cdot f(a)}{f'a}}$.

$v$ is the gene vel. in the eccen.

---

$V$ = vel? in $\odot$
$v$ = vel? at gi. apse

$v : V : G : f$

$V = \frac{f}{G} \cdot v$

$\frac{f^2 \cdot v^2}{G^2} = V^2 = \frac{2\mu}{r} - \frac{\mu}{a_1}$

$= \mu \cdot \left(\frac{2}{a_1(1+\varepsilon)} - \frac{1}{a_1}\right)$

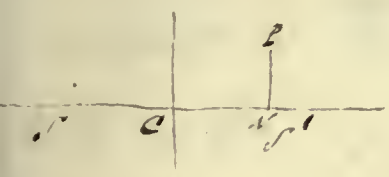
$= \frac{\mu \cdot (1-\varepsilon)}{a_1(1+\varepsilon)} = \frac{\mu(1-\varepsilon)}{a}$

$\frac{G}{f} = \sqrt{\frac{f(a)}{a f'(a)}}$.

$\therefore \frac{\mu \cdot (1-\varepsilon)}{a} = v^2 \cdot \frac{f(a)}{a \cdot f(a)}$

$v^2 = \mu \frac{f'(a)}{f(a)} \cdot (1-\varepsilon)$

the velocity of the gene body at the apse. (Newton Prop 45) being known. fix the eccen. of the $\odot$ employed in the propositn.




i.e. the centre of force. $\therefore x_1 = 2c$

Let the middle point of $ii$ be taken for the origin of coords. & the points of the pendulum the ground. a.g. the coords of $P$.

$$\text{Then } \frac{d^2u}{t} = \mu \left\{ \frac{a+u}{(a+u)^2+y^2} - \frac{a-u}{(a-u)^2+y^2} \right\}.$$

$$\frac{2}{t}\frac{du}{t}\frac{d^2u}{t} = \mu \cdot \frac{2}{t}\left\{ \frac{1}{2}\frac{(a+u)^2+y^2}{(a+u)^2+y^2} + \frac{1}{2}\frac{(a-u)^2+y^2}{(a-u)^2+y^2} \right\}$$

$$\therefore \left(\frac{du}{t}\right)^2 + \mu \log\{(a+u)^2+y^2\}\{(a-u)^2+y^2\}.$$

But this by the question is constant.

$$\therefore \{(a+u)^2+y^2\}\{(a-u)^2+y^2\} = c^4$$

which is the $=^n$ to the Lemniscate.

Determine the value of $\frac{d^2}{d\theta^2}+u$ when expressed in terms of the radius of curvature.

Let $\rho$ be the radius of curv.

$$\therefore \rho = \frac{1}{u^3}\cdot\frac{\left\{\left(\frac{du}{d\theta}\right)^2+u^2\right\}^{\frac{3}{2}}}{\left(\frac{d^2u}{d\theta^2}+u\right)}$$

$$\text{Therefore } \frac{d^2u}{d\theta^2}+u = \frac{1}{u^3}\left(\left(\frac{du}{d\theta}\right)^2+u^2\right)^{\frac{3}{2}}\frac{1}{\rho}.$$

Now when $\frac{du}{d\theta}=0$.

$$\therefore \frac{d^2u}{d\theta^2}+u = \frac{1}{\rho}$$

A body revolves in an ⊃ the centre of forces being the focus. Then that $h = \frac{2\pi}{T} \sqrt{\mu a.(1-e^2)}$.

$$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}$$

at the greater apse. the direction of the body is ⊥ to the radius vector. and the radius vector $= a(1+e)$. also the velocity at this point $= \sqrt{\frac{\mu}{a}\left(\frac{2}{1+e} - \frac{1-e}{1+e}\right)}$

$$= \sqrt{\frac{\mu}{a} \cdot \frac{1-e}{1+e}}.$$

$$\therefore h = vp = a.(1+e)\sqrt{\frac{\mu.(1-e)}{a(1+e)}}$$

$$= \sqrt{\mu.a(1-e^2)}.$$

$$\text{also } T = \frac{2 \text{ area of } \odot}{h} = \frac{\pi.ab}{\sqrt{\mu.a(1-e^2)}}.$$

Now $b = a\sqrt{1-e^2}$.

$$\therefore T = \frac{\pi.a^{\frac{3}{2}}\sqrt{a(1-e^2)}}{\sqrt{\mu}\sqrt{a(1-e^2)}} = \frac{\pi a^{\frac{3}{2}}}{\sqrt{\mu}}$$

Divide a vertical space into 3 parts such that the time of descent down each may be the same.

$$s = \frac{1}{2} f t^2$$

Changing $t$ into 1. 2. 3 successively, and denoting by $s$. $s_1$ $s_2$ the spaces through which the body has fallen at the end of these periods we have $s = \frac{1}{2} f$, $s' = \frac{4}{2} f$, $s'' = \frac{9}{2} f$

∴ the spaces descended are as 1 : 4 : 9 —

wherefore if $a$ = the whole vertical space which a body descends a given space of time, and it be divided into the spaces

$\frac{a}{9}$, $\frac{3a}{9}$, $\frac{5a}{9}$, the times of descent down each will be the same.

A body is projected upwards with a given velocity, find the space which it has described after a given time.

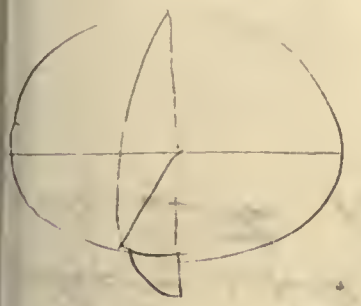
Let $s$ = space described, $t$ the time &c —

Then $s = vt - \frac{1}{2} f t^2$ =

Solving the eq. with respect to $t$

$$t = \frac{v}{f} \left\{ 1 \pm \sqrt{v^2 - 2sf} \right\}$$

The double result corresponding to the two values of the sign before the radical refers to the time of passing the same point of the curve, in the body moving upwards, and when falling.



$Ax^2 + By^2 + Cz^2 = 1.$

Let $x = mz$ $y = nz$ ($l, m, n$ the dirs to a diameter

$$A(mz+h)^2 + B(kz+k)^2 + C(nz)^2 = 1,$$

must give = l values of $z$ with diff ratios

or $\quad 2Amh + 2Bkk + 2Cl = 0$

the = n to the diametral plane.

or $Amx + Bny + Cz = 0$ suppose. (1)

$Am_1x + Bn_1y + Cz = 0$ (2) is the = n to the

diametral plane of $\begin{matrix} x = m_1 z \\ y = n_1 z \end{matrix}$ (2)

$Am_2x + Bn_2y + Cz = 0$ (3) — of $\begin{matrix} x = m_2 z \\ y = n_2 z \end{matrix}$ (3)

But the diametral sections of (1) & (2) must coincide with (3) in order that (1), (2), (3) may be conjugate diameters

$$A(mn_1 - m_1 n)x + C(n_1 - n) = 0$$

$$\therefore \quad x = -\frac{C(n - n_1)}{A(mn_1 - m_1 n)} z = m_2 z$$

$$y = \frac{C(m - m_1)}{A(n_1 m - m_1 n)} z = n_2 z$$

$$\& \quad \frac{C(n - n_2)}{A(mn_2 - m_2 n)} = m_1$$

$$\frac{C}{A}\frac{(n - n_2)}{(n_2 m - m_2 n)} = n_1$$

A body moved in a curve acted upon by a force // to the axis of y. Prove that the force $= u^2 \cdot d^2_x y$ where u is the velocity // to the axis of x. and apply this $= a$ to determine the path of a projectile upon the earths surface.

$$d^2_t y = F. \qquad d^2_t u = 0.$$

$$d_t u = c. = u.$$

Now $d_t y = d_x y \cdot d_t x = u \cdot d_x y.$

$$d^2_t y = d_t \left( d_t y \right) = d_t \left\{ u \cdot d_x y \right\} = d_x \left\{ u \cdot d_x y \right\} d_t x$$

$$= u \cdot d^2_x y \cdot u$$

$$= u^2 \cdot d^2_x y.$$

$$\therefore F = d^2_t y = u^2 \cdot d^2_x y.$$

For the case of a projectile.

$$u^2 d^2_x y = -g.$$

$$\therefore d^2_x y = -g \theta^2 \sec^2 \alpha. \quad \left\{ \text{where } \alpha = \text{angle of proj.} \right.$$

$$\therefore d_x y = -gx \cdot \theta^2 \sec^2 \alpha + c.$$

$$x = 0 \quad d_x y = \tan \alpha.$$

$$\therefore d_x y = \tan \alpha - gx \cdot \theta^2 \sec^2 \alpha$$

$$y = x \tan \alpha - \frac{gx^2}{2v^2} \sec^2 \alpha.$$

A heavy body is projected along the interior surface of a cylinder the axis of which is vertical determine the path.

Let the point of projection be taken for the origin of coords.

Let the centre of the horizontal circular section of the ~~which the~~ cylinder which passes thro' the point of projection be taken for the origin of coords.

Let $\alpha$ be the direction of projection with respect to the horizontal and $v$ is the constant horizontal velocity ∴ $v\cos\alpha$ is the constant angular velocity estimated horizontally.

But since the body is subject to the action of gravity if $z$ be the vertical coord. of the body at the time $t$. $z = gt - \frac{1}{2} vt\sin\alpha - \frac{1}{2} gt^2$.

Therefore for the coords at any time

$$x^2 + y^2 = r^2.$$

$$z = vt\sin\alpha - \frac{1}{2} gt^2.$$

or if $\theta$ be the angle described by the radius vector estimated horizontally

$$\theta = \frac{tv\cos\alpha}{r}.$$

$$z = vt\sin\alpha - \frac{1}{2} gt^2$$

I find the attn upon uniform cylindrical rod, the law of which is very small, in the direction of its length upon a point any where without it.

AB the cylindrical rod.
k the area of a vertical section of the rod. P the point. a. — c the const of P & y. & x. axes of x.




& v = x the the element of rod = k. δx. the attn of this element on the pt = $\frac{k\rho\,\delta x.}{Pv^2} = \frac{k\rho\,\delta x}{a^2+(c+x)^2}$ in direction Pv.

∴ the attn in direction of the length of the rod = $\frac{k\rho\, a.\,\delta x}{\{a^2+(c+x)^2\}^{\frac{3}{2}}}$.

∴ if A = whole attn.

$$A = \int_{x.}^{0.B.} \frac{k\rho\, a.}{(a^2+(c+x)^2)^{\frac{3}{2}}}$$

$$= \frac{k\rho}{a} \cdot \frac{c+x}{\{a^2+(c+x)^2\}^{\frac{1}{2}}} + C$$

$$0 = \frac{k\rho}{a} \cdot \frac{c}{a} + C.$$

$$\therefore A = \frac{k\rho}{a}\left\{\frac{c+b}{\{a^2+(c+b)^2\}^{\frac{1}{2}}} - \frac{c}{a}\right\}.$$

The resistance of the air varying as the square of the velocity determine the motion of a body falling from rest, and shew that after a certain finite time the velocity becomes nearly uniform

$$\frac{d^2x}{dt^2} = g - kv^2. \quad \text{or} \quad \frac{dv}{dt} = g - kv^2.$$

$$\therefore dt = \frac{1}{g - kv^2} = \frac{1}{k} \cdot \frac{1}{\frac{g}{k} - v^2}.$$

$$\therefore t = \frac{1}{2\sqrt{kg}} \cdot \log \left( \frac{\sqrt{g} + v\sqrt{k}}{\sqrt{g} - v\sqrt{k}} \right)$$

$$\therefore e^{2t\sqrt{kg}} = \frac{\sqrt{g} + v\sqrt{k}}{\sqrt{g} - v\sqrt{k}}.$$

$$\left(\sqrt{g} - v\sqrt{k}\right) e^{2t\sqrt{kg}} = \sqrt{g} + v\sqrt{k}.$$

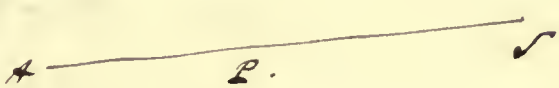
$$\sqrt{g}\left\{e^{2t\sqrt{kg}} - 1\right\} = v\sqrt{k}\left\{1 + e^{2t\sqrt{kg}}\right\}$$

$$\therefore v = \sqrt{\frac{g}{k}} \cdot \left( \frac{e^{2t\sqrt{kg}} - 1}{e^{2t\sqrt{kg}} + 1} \right)$$

which is nearly uniform after a certain finite time $t$.

A body attracted to a constant centre of force moves directly towards it from rest through a medium of which the resistance $= \frac{n}{2}\left(\text{vel}^y\right)^2$ : find the velocity acquired through a given space and determine the diff. from the centre when the velocity is a maximum.

+ the point from which the body starts. $+S = a$. $P$ the position of the body at a time $t$. $PS = x$.




For the motion

$$\frac{dv}{dt} = c - \frac{nv^2}{2x} \quad \text{or} \quad \frac{dv}{dx} = c\frac{dt}{dx} - \frac{nv}{2x} \quad \text{for } v = \frac{dx}{dt}.$$

$$\therefore \frac{dv}{v} = c\frac{dt}{dx} - \frac{n}{2x}.$$

$$d_x(\log v) = c$$

For the motion $v\frac{dv}{dx} = c - \frac{n}{2}\frac{v^2}{x}$.

$$v\frac{dv}{dx} + \frac{n}{2}\frac{v^2}{x} = c.$$

$$d_x\left(e^{n\log x}\cdot v^2\right) = 2ce^{n\log x}.$$

$$e^{n\log x}v^2 = 2c\int x^n e^{n\log x}.$$

$$= 2c.xe^{n\log x} - 2cn\int e^{n\log x}$$

$$= \frac{2cx}{1-n}.e^{n\log x} + C.$$

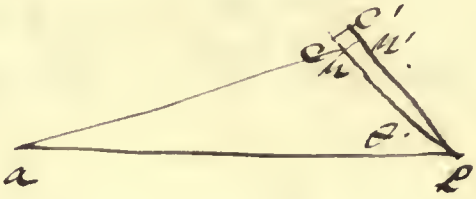
$$0 = \frac{a}{1-n}.e^{n\log a} + C.$$

$$\therefore e^{n\log x}.v^2 = \frac{2c}{1-n}\left\{x^n e^{n\log x - n\log a}\right.$$

$$v^2 = \frac{2c}{1-n}.\left(x - ae^{\log\frac{a}{x}}\right)$$

Find the attraction of a sphere on a particle within it, the force varying inversely as the square of the dist.

$P$ the centre of the sphere.
$a$ the point at which the particle is placed. $Pa = b$. $c$ = rad of the sphere.




$PC = PC' = r$. $CPa = \theta$. $C'Pa = \theta + \delta\theta$. $aA = u$.

$u^2 = b^2 + r^2 - 2br\cos\theta$ $\therefore 2u\,du = 2br\sin\theta\,d\theta$ (1)

area of the elem. $C'A = r\,\delta\theta . \delta r$.

vol of the annulus generated by the revolution of this annulus about the line $aP$ $= 2\pi r^2 \sin\theta\,\delta\theta\,\delta r$.

$= \frac{2\pi ur}{b} . du\,\delta r$ from (1)

$\therefore$ attn of this annulus in direct. $Pa = \frac{2\pi\rho}{b}\frac{r}{u} . du\,\delta\theta\,\delta r \sin\theta$

$= \frac{\pi\rho}{b^2} \cdot \int r \int_{b-r}^{b+r} \left(\frac{u^2 + b^2 - r^2}{u^2}\right) r$

$\int_{b-r}^{b+r} du \cdot r\left(\frac{b^2 - r^2}{u^2}\right) + r = -r(b^2 - r^2)\left(\frac{1}{b+r} - \frac{1}{b-r}\right) + r\{b+r - b+r\}$

$= -r\{b+r - b+r\} + r(b+r - b+r)$

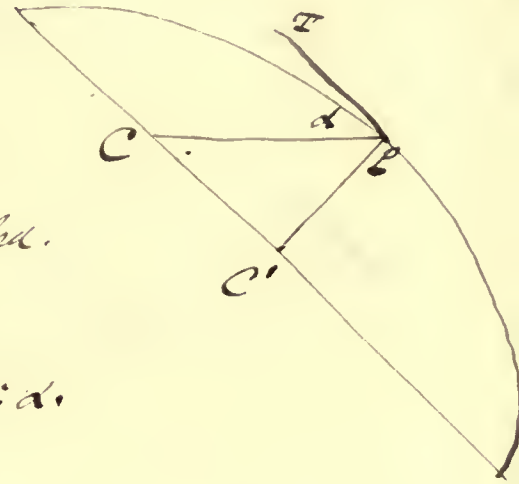
$= +4r^2$

and $\int_0^c 4r^2 = \frac{4}{3}c^3$.

$\therefore$ attn of the sphere $= \frac{4}{3}\cdot\frac{\pi\rho c^3}{b^2}$

$= \frac{\text{mass of the sphere}}{b^2}$.

Force $= \frac{\mu}{SP^2}$. At a distance $SL$ from the centre bodies are projected in all directions with a velocity $= \sqrt{\frac{2\mu}{SL}}$. Then that the axis major of any of the ellipses described will be parallel to the direction of projn. and their centres will all lie in the surface of a sphere of which $SL$ is a diameter.




C the cent. of force. L the pt. in $LT$ the direct. of velocity of the bodies.

then $S = \frac{v^2}{2E} = \frac{SL}{2}$.

Let $a.b$ be semi axes of the orbit described.

$$2a = \frac{S^2}{r-S} = 2 \therefore SL \therefore a = SL.$$

$$b = \frac{S}{r-S} \cdot r\sin\alpha = r\sin\alpha = SL\sin\alpha.$$

$$\therefore e^2 = 1 - \frac{b^2}{a^2} = 1 - \sin^2\alpha = \cos^2\alpha.$$

If $\theta$ = ang. cent. of $LT$ to the axis ~~major~~ minor

$$\sin\theta = \frac{1}{e}\cos\alpha = 1 \therefore$$ the axis minor is $\perp$ to the direction of projn. $\therefore$ axis major is $\parallel$ to the direction of projn.

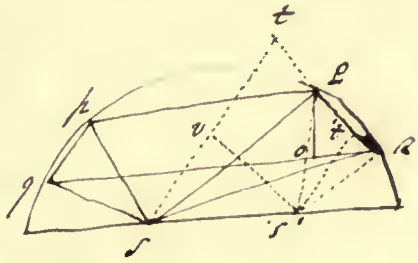
Also $CC' = CL\cos\alpha$. and as $CC'L$ is a right angle $C'$ is a point in the sphere of which $CL$ is the diameter, and the plane is true of all the other orbits $\therefore$ all the orbits have their cent. in that sphere, and their axes major $\parallel$ to the direct. of projn.

In an ⊙ described round a force in one of the foci two chords $Qp$, $Rq$ are drawn ∥ to the axis major, shew that the difference of the times in which the arcs $QR$, $pq$ are described varies as the distance between the chords.

Let $S$ be the centre of force.

$S'$ the other focus of the ⊙

$St$, $S't'$ ⊥ on the chord $QR$.




Now the times of describing the arcs will be ∴ ∝ to the areas swept out by the radii vectores.

also area $Spq$ = area $S'QR$.

∴ the difference of the areas = area $QSR$ − area $QS'R$.

$$= \tfrac{1}{2} QR \cdot (St - S't')$$

$$= \tfrac{1}{2} QR \cdot Sv.$$

Now by simlr ∆s $SS' : Sv :: QR : QO$

or $QO \times SS' = QR \cdot Sv$

∴ the difference between the areas varies as $QO \cdot SS'$

∠ $QO$ for $SS'$ is constant

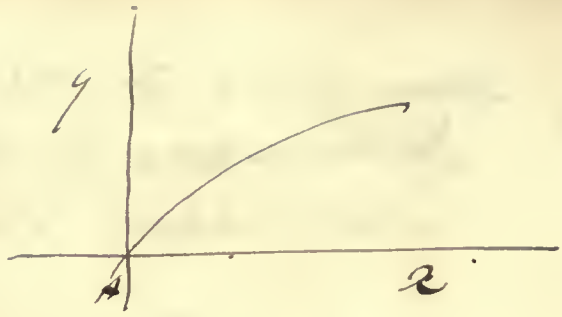
but the difference of the areas is ∴ ∝ the difference of the times of describing the arcs

∴ the diff of the time in describing arcs is ∴ ∝ to $QO$. the dist. between the chords.

$$\frac{d^2x}{t} = \frac{dx}{s} \cdot 2g \cdot \sqrt{\frac{v}{h}}.$$

$$= \frac{dx}{s} \cdot 2g \cdot \sqrt{\frac{ds}{h}}.$$

$$\therefore \left(\frac{dx}{t}\right)^2 = \left(\frac{dx}{s}\right)^2 \frac{4g^2}{h} \frac{ds}{t} = \frac{4g^2}{h} \cdot \frac{dx}{t}.$$




$$\frac{dx}{t} = w.$$

$$\therefore \frac{dw}{t} = \frac{2g}{h} \sqrt{w}. \quad , \quad \frac{dt}{w} = \frac{h}{2g} \frac{-1}{\sqrt{w}} \quad \therefore t = \frac{\sqrt{h}}{g}\left(\sqrt{w} + \sqrt{a}\right)$$

$$\therefore \sqrt{w} = \left(\sqrt{a} - \frac{gt}{\sqrt{h}}\right). \quad \frac{dx}{t} = \left(\sqrt{a} - \frac{gt}{\sqrt{h}}\right)^2.$$

Similarly $\frac{dy}{t} = \left(\sqrt{a_1} - \frac{gt}{\sqrt{h}}\right)^2.$

$$\text{But } v^2 = \left(\frac{dx}{t}\right)^2 + \left(\frac{dy}{t}\right)^2 = \left(\sqrt{a} - \frac{gt}{\sqrt{h}}\right)^4.$$

where $a$ is some constant.

$$\therefore v = \left(\sqrt{a} - \frac{gt}{\sqrt{h}}\right)^2.$$

A body is projected in a medium where the resistance $= 2g \cdot \frac{v}{h}$. Shew that the velocity after any time $= \left(\sqrt{a} - \frac{gt}{\sqrt{h}}\right)^2$.

$E = \mu\left\{r + \frac{2a^3}{r^2}\right\}$ find with what velocity a body must revolve at a distance $a$, and if the velocity were suddenly doubled shew that the new orbit would have an apse at distance $3a$.

If $r = a$ $E = 3\mu a$. Now in a $\odot$ $v = \sqrt{E.R}$

$\therefore$ In this case the velocity $= a\sqrt{3\mu}$.

also $h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = \frac{\mu}{u} + 2a^3\mu.u^2$

or $\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}\frac{1}{u^3} + \frac{2a^3u}{h^2}$.

But $h^2 = a^2v^2 = 3\mu a^4 \times 4 = 12\mu a^4$.

$\therefore \frac{d^2u}{d\theta^2} + u = \frac{1}{4.3a^4}\frac{1}{u^3} + \frac{2}{12a}$.

$\therefore \left(\frac{du}{d\theta}\right)^2 + u^2 = -\frac{1}{12a^4}\frac{1}{u^2} + \frac{u}{3a} + C$.

at the time the velocity was doubled

$\frac{du}{d\theta} = 0 \quad u = \frac{1}{a}$.

$\therefore \frac{1}{a^2} = -\frac{1}{12a^2} + \frac{1}{3a^2} + C$.

$\frac{12 + 1 - 4}{12a^2} = \frac{3}{4a^2} = C$.

$\therefore \left(\frac{du}{d\theta}\right)^2 = -u^2 - \frac{1}{12a^4}\frac{1}{u^2} + \frac{u}{3a} + \frac{3}{4a^2}$

at an apse $\frac{du}{d\theta} = 0$.

$\therefore u^2 + \frac{1}{12a^4}\frac{1}{u^2} = \frac{u}{3a} + \frac{3}{4a^2}$.

which is satisfied if $u = \frac{1}{3a}$.

$\therefore$ at the apse $r = 3a$.

When a body describes an ellipse about a centre of force in the centre, it moves between the extremity of any pair of conjugate diameters in the same time.

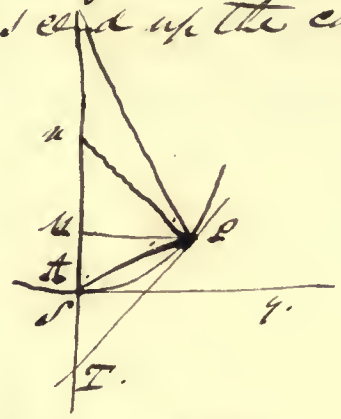
For the $\frac{1}{2}$ ellipse $\odot$ referred to any conjugate diameters as axes is $y^2 = \frac{b'^2}{a'^2}(a'^2 - x^2)$.

$\therefore$ the area contained between any semi-conjugate axis $= \frac{1}{4}$ area of the $\odot$

wherefore since the radius vector describes ~~the~~ equal areas in equal times, and the centre of force is at the origin of coord$^{ts}$, the radius vector will describe with the space between any semi conjugate diameters in $\frac{1}{4}$ of the whole periodic time

$\therefore$ the body moves between the extremities of any two conjugate diameters in the same time

A centre of force which is repulsive and varies as dist$^{n}$ is situated at S the lowest point of a vertical O/f from for will a body placed at A and acted on by gravity at end of the canal.




Let P be the position of the body at the time t

SM = x. MP = y coords of P.

The force of S upon P = $\frac{\mu}{SP^n} = \frac{\mu}{r^n}$ suppose in

direction SP = $\frac{\mu x}{SP^{n+1}}$ in direction SM.

$\tan PMT = \frac{dy}{dx} = \frac{a-u}{\sqrt{2au-u^2}}$.

$\tan MSP = \frac{y}{x} = \frac{\sqrt{2au-u^2}}{u}$ $\therefore \tan SPT = \frac{\frac{\sqrt{2au-u^2}}{x} - \frac{a-u}{\sqrt{2au-u^2}}}{1 + \frac{a-u}{x}}$.

the force exerted by S along the tangent =

$\frac{\mu}{r^n} \cdot \frac{\sqrt{2a-x}}{\sqrt{2a}}$.

$= \frac{(2au-u^2) - ax + x^2}{a\sqrt{2au-u^2}}$

$= \frac{x}{\sqrt{2au-u^2}}$

$\therefore \cos SPT = \frac{\sqrt{2au-u^2}}{\sqrt{2ax}}$

$= \sqrt{\frac{2a-x}{2a}}$.

the whole force along the tangent =

$g \frac{\sqrt{2au-u^2}}{a} - \frac{\mu}{(2ax)^{\frac{n}{2}}} \cdot \frac{\sqrt{2a-x}}{\sqrt{2a}}$.

$v\,dv = -\frac{\mu}{r^n}\,dr + g\,dx$

$\frac{1}{2}v^2 = \frac{\mu}{n-1}\frac{\mu}{r^{n-1}} + gx + C$.

$x = 0$ $r = 0$ $v = 0$ $\therefore C = 0$.

$\frac{1}{2}v^2 = \frac{\mu}{n-1}\frac{1}{(2ax)^{\frac{n-1}{2}}} + gx$.

when $v = 0$ we have.

$0 = \frac{\mu}{n-1} \cdot + (2a)^{\frac{n-1}{2}} g \cdot x^{\frac{n+1}{2}}$

$\therefore x =$

$\phi = \mu u + \mu' u^2$ the apsidal angle $= \frac{\pi\left(1 + \frac{\mu'}{h^2}\right)}{\sqrt{1 + \frac{\mu'}{h^2}}} = \pi\sqrt{1 + \frac{\mu'}{h^2}}$

From this then that the above value of $\phi$ is coincident with that given by Newton for a body moving in a revolving orbit

$$\frac{8A^2}{4.Cp^2} + \frac{G^2 - E^2}{E^2} \cdot \frac{4A^2}{Ch^3}$$

we have the apsidal angle $= \frac{G}{E}\pi$.

$\therefore \frac{G}{E} = \sqrt{1 + \frac{\mu'}{h^2}}$ $\quad \frac{G^2 - E^2}{E} = + \frac{\mu'}{h^2}$.

also $8A^2 = 2h^2$ and $4A^2 = h^2$.

$\therefore$ the expression becomes $\left(Cp = r = \frac{1}{u}\right)$

$$\frac{2h^2}{C} \cdot u^2 + \frac{\mu'}{h^2} \cdot \frac{h^2}{r^3}$$

$$= \mu u^2 + \mu' u^3 \quad \text{for } \frac{2h^2}{C} = \mu.$$

Shew that Newtons expression for the force

$$= \frac{2h^2}{\int X^2 . \rho v}$$

agrees with the analytical expression $P = h^2 u^3 \left(\frac{d^2 u}{d\theta^2} + u\right)$

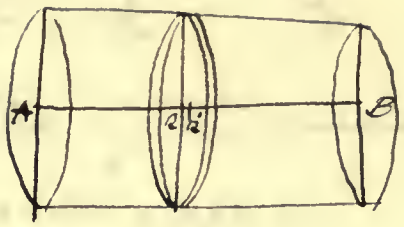
In this case $\int X^2 = \frac{1}{u^2 + \left(\frac{du}{d\theta}\right)^2}$.

Also $\rho v = \frac{1}{2.u^2} \cdot \frac{u^2 + \left(\frac{du}{d\theta}\right)^2}{u + \frac{d^2 u}{d\theta^2}}$.

$\therefore \frac{2h^2}{\int X^2 . \rho v} = \frac{h^2 u^2}{\quad} \left(\frac{d^2 u}{d\theta^2} + u\right)$.

Shortly attached to a contract

Let $a$ = rad of the base of the cylinder. $b$ = its height.




Take AB the axis of the cylinder. for the axis of $x$. Then the mass of a section contained between two planes $\perp$ to the axis of $x$, and at distances of $x$. $x+\delta x$ from A will be.

$$\pi \rho \frac{a^2}{2} \delta x.$$

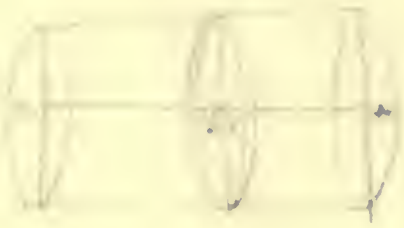
$\therefore$ if $M$ = mass of the cylinder. and $k$ rad of gyn.

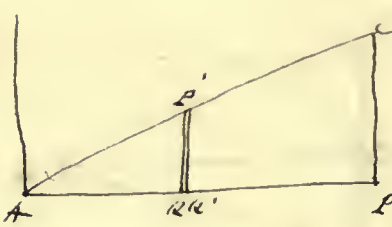
$$M k^2 = \pi \rho \frac{a^4}{2} \int_0^b dx \quad \text{from } x = 0 \text{ to } x = b$$

$$= \frac{\pi \rho a^4 b}{2}$$

But $M = \pi \rho a^2 b$

$$\therefore k^2 = \frac{a^2}{2}.$$






let AP the axis of the cone be the axis of x. $y = mx$ the eqn to the generating line AB.

Then the mass of a thin circular sect. generated by the revolut. of PRR' about the axis of x ($AR = x$ $AR' = x + \delta x$) $= \rho \delta x . \pi y^2$

and the moment of inertia about the axis of x =

$$\tfrac{1}{2} \pi \rho y^4 \delta x .$$

∴ the moment of inertia of the whole cone

$$= \tfrac{1}{2} \pi \rho \int_0^{AP} x . y^4 \quad . = \tfrac{1}{2} \pi \rho m^4 \int_0^b x^4 \quad \text{if} \quad AP = b .$$

$$= \frac{1}{2} \frac{\pi \rho m^4 b^5}{5} .$$

∴ if $k$ = rad of gyn mchane.

$$M k_2 = \frac{\pi \rho m^4 b^5}{2.5.}$$

But mass of the cone $= \dfrac{m^2 b^3 \pi \rho}{3.}$

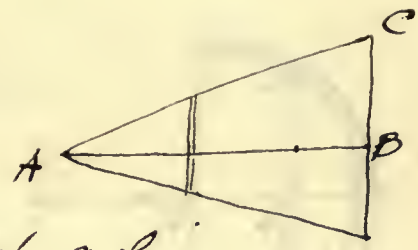
$$\therefore k^2 = \frac{3}{2.5} . m^2 b^2$$

$$= \frac{3}{2.5} BP^2 = \frac{3}{10} BP^2 .$$



AB the axis of the cone. = b

a = rad of the base.




(1) To find the moment of inertia about a line thro' A ⊥ to the axis of the cone.

The mass of a slice of the cone thickness. δx. cut off by planes ⊥ to the axis at distances x. x+δx from A will be.

$$\frac{\pi\rho a^2}{b^2}\cdot x^2\,\delta x.$$

& the moment of inertia of this slice about any diam will be.

$$\frac{\pi\rho}{4}\frac{a^4}{b^4}x^4\,\delta x \quad \therefore \quad \frac{(\pi\rho a^2)(a^2+4b^2)}{4b^4}x^4\,\delta x = \text{Moment of inertia}$$

about an axis thro' A ⊥ to AB.

∴ if M = mass of the cone & k rad of gyn. about this axis

$$Mk^2 = \frac{\pi\rho a^2}{4b^4}\cdot(a^2+4b^2)\int_0^b x^4$$

$$= \frac{\pi\rho a^2 b}{20}(a^2+4b^2).$$

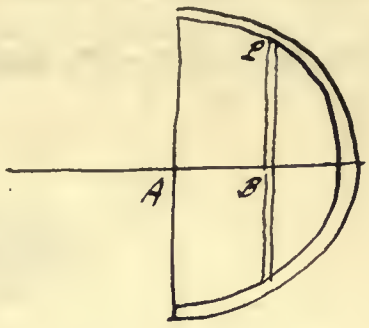
But $M = a^2\pi\cdot\frac{b\rho}{3}$ ∴ $k^2 = \frac{3}{20}\cdot(a^2+4b^2)$ —

wherefore if $k_1$ = rad of gyn. about an axis thro' the cent. of grav. ⊥ to the axis of the cone.

$$k_1^2 = \frac{3a^2}{20} + \frac{3.4}{20}b^2 - \frac{4}{9}b^2 = \frac{27a^2-28b^2}{180}.$$



A the cent. of the sphere.
a. b. the radii of the outer and inner spheres.




The moment of inertia of the annulus contained between planes ⊥ to the axis at a distance x and x + δx from A will be.

$$\frac{\pi\rho.(a^4-b^4)\,\delta x}{2.} = \frac{\pi\rho}{2}.\left(a^2-x^2\right)^2-\left(b^2-x^2\right)^2\delta x.$$

∴ Moment of inertia of the hollow sphere =

$$\pi\rho\int_b^a a^4-2ax^2+x^4-b^4+2bx^2+x^4)$$

or $Mk^2 = \pi\rho\left\{a^5-\frac{2}{3}a^5+\frac{1}{5}a^5-b^5+\frac{2}{3}b^5-\frac{1}{5}b^5\right.$

$$= \frac{\pi\rho 8}{1/5}\left\{a^5-b^5\right)$$

But $M = \frac{4}{3}\pi\rho(a^3-b^3)$

$$\therefore k^2 = \frac{2}{5}.\left(\frac{a^5-b^5}{a^3-b^3}\right)$$



Find the velocity with which a body must move near the earth's surface in order to continue revolving about it. In a circular orbit.

$$L = 2\pi \sqrt{\frac{R}{F}} = 2\pi \sqrt{\frac{3960 \times 5280}{32.2}}.$$

$$= 2\pi \sqrt{\frac{660 \times 6 \times 660 \times 8}{32.2}} = 660 \times 8 \times \pi \sqrt{\frac{3}{32.2}}$$

Now $\sqrt{\frac{3}{32.2}} = \sqrt{.0931677} = .30523.$

$$\therefore L = \pi \times 5280 \times .30523$$

$$= 3.1416 \times 5280 \times .30523$$

$$= 50623 \text{ seconds nearly}$$

$$= 1^{hr}\ 24'\ 23''$$

The force tending to the centre of a ⊙ whose radius is $c$, being $\mu\left(r + \frac{2c^4}{r^3}\right)$ find the velocity with which a body will describe the ⊙ and then that if the velocity be suddenly doubled it will come to an apse at a distance $c$.

1st. to find the velocity.

$$v = \sqrt{E.R} = \sqrt{\mu r^2 3} = r\sqrt{3\mu}, \text{ for } r = c.$$

let $h$ = 2 area described in 1 second, in the curve after the velocity is doubled.

$$h^2 = 4v^2.r^2 = 12c^4\mu.$$

∴ since the force is central

$$h^2u^2.\left(\frac{d^2u}{d\theta^2} + u\right) = \mu.\frac{1}{u} + 2c^4u^3$$

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{12c^4}\left\{\frac{1}{u^3} + 2c^4u\right)$$

$$\frac{d^2u}{d\theta^2} = \frac{1}{12c^4}\left\{\frac{1}{u^3} - 10c^4u\right\}.$$

$$\left(\frac{du}{d\theta}\right)^2 = \frac{1}{12c^4}\left\{-\frac{1}{u^2} - 10c^4u^2\right) + C.$$

at the point of apsis. $\frac{du}{d\theta} = 0$ $u = \frac{1}{c}$.

$$0 = \frac{1}{12c^4}\left\{-11c^2\right) + C.$$

$$\therefore \left(\frac{du}{d\theta}\right)^2 = \frac{1}{12c^4}\left\{-\frac{1}{u^2} - 10c^4u^2 + 11c^2\right\}.$$

$\frac{d^2u}{d\theta^2} + u = \frac{1}{r^2}$

$\therefore 2u = \frac{v^2}{\mu r} = \frac{12\mu c^2}{6\mu c} = 2c.$

$2u = \frac{4c^4}{c-2c} = -4c$

A spherical surface is described in space, having in its centre a force $\propto \frac{1}{\text{dist}^2}$. Then that if a particle be let fall from this surface, and be projected in any direction at any moment of its descent with the velocity acquired in falling to the moment on ⊙ the major axis of which = the radius of the sphere.

To find the velocity acquired.

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2} \quad \therefore \left(\frac{dx}{dt}\right)^2 = 2\frac{\mu}{x} - \frac{\mu}{a} = \frac{2\mu}{x}\left(\frac{a-x}{a}\right).$$

$$= \frac{\mu}{x}\left(2 - \frac{x}{\frac{1}{2}a}\right). \quad (1)$$

Now in a ⊙ $F = \frac{\mu}{x^2}$ whence

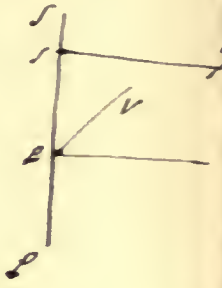
$$v^2 = \frac{\mu}{x}\left\{2 - \frac{x}{AC}\right\}. \quad (2)$$

Compare (2) with (1) and we see they are similar, and that $AC = \frac{1}{2}a$, or $AA'$ the major axis of the ⊙ $= a$

two masses S. L. S being = 2L attract each other with force $\frac{2}{(dist)^2}$ and
would towards each other in a straight line from rest. when they have approached
by ½ of their original distance L receives a new velocity ⊥ to their motion, and
double of that which S has acquired — what curve will each now describe
described by each about the other.

Since S = 2L, at the moment attraction ceases the velocity
of L = twice the velocity of S. Also S will evidently have moved through ⅓ of the
whole space. ∴ $v^2 = \left(\frac{dx}{dt}\right)^2 = 2\mu\left\{\frac{1}{x} - \frac{1}{3x}\right\} = \frac{2\mu}{3a}$. = velocity of S

$v_1^2 = \frac{4\mu}{3a}$ = velocity of L




Let a force equal to that which L exerts upon S
be applied to both L and S in opposite direction to
that in which S had tendency to move. This, since, work
done the relative position of S and L. as if neither had moved.

Let SL be the axis of x. Sy ⊥ to SL the axis of y. also let x, y. be
the coords of L at any time t.

Then $\frac{d^2x}{dt^2} = -\frac{(S+L)}{r^3}x$. $\frac{d^2y}{dt^2} = -\frac{(S+L)}{r^3}y$.

Or using polar coords as the force is central.

$h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = (S+L)u^2 = \mu u^2$ suppose. (3)

Now if $v^2$ = velocity at L $v^2 = \frac{6\mu}{5a}$ $\left(= v^2 + v_1^2\right)$.

also $\frac{1}{2}SL^2\dot{\theta} = h = \frac{1}{2}$ ∴ $SL^2\dot{\theta} = 1$

∴ $h^2 = SL^2 \sin^2 SLV \cdot v_1^2 = \frac{a^2}{4}\cdot\frac{6\mu}{25a} = \frac{3a\mu}{50}$.

∴ eqn (3) becomes. $\frac{d^2u}{\theta^2} + u = \frac{\mu}{h^2} = \frac{50}{3a}$.

∴ $u = \frac{50}{3a} + A\cos(\theta - B)$ which compared with $u = \frac{1}{a(1-e^2)}\{1+e\cos(\theta-B)\}$

gives $\frac{50}{3a} = \frac{1}{a(1-e^2)}$. $A = \frac{e}{a(1-e^2)} = \frac{50}{3a}e$ ∴ $A$ = ecc the major axis —

may be found.

wherefore L describes a Δ about S,

similarly it may be shown that S describes a Δ about L, and
that the two bodies describe ellipses about each other.

$$h^2 u^2 \left(\frac{d^2u}{\theta^2} + u\right) = \mu u^3 + \mu' u^2$$

$$\frac{d^2u}{\theta^2} + u = \frac{\mu}{h^2} + \frac{\mu' u}{h^2}.$$

$$\frac{d^2u}{\theta^2} + \left(1 - \frac{\mu'}{h^2}\right) u - \frac{\mu}{h^2} = 0.$$

$$\therefore u = \frac{\mu}{h^2} \cdot \frac{h^2}{h^2 - \mu'} + A \cos\left(\frac{h^2}{h^2 - \mu'} \theta - B\right)$$

$$= \frac{\mu}{h^2 - \mu'} + A \cos \frac{h^2}{h^2 - \mu'} \left(\theta - B'\right) \}$$

Now in an ⊙ $u = \frac{1}{a(1-e^2)} + \frac{e}{a(1-e^2)} \cos(\theta - B')$

∴ the proposed orbit will be an ellipse, the velocity of the plane of the ⊙ being to that of the radius vector as $\frac{\mu'}{h^2 - \mu'}$ is to unity.

The central force consists of two parts, of which one varies as $\frac{1}{(\text{dist.})^2}$ and the other as $\frac{1}{(\text{dist.})^3}$. When that the orbit described will coincide with the path of the body revolving in an ⊙ round a cent. of force in a focus, while the plane of the ⊙ revolves at the same time about that centre.

A heavy ball attached by a string to a fixed point is held in such a position that the string is horizontal, find the tension of the string at the lowest point.

For the tension of the string we have. $R = \frac{mv^2}{\rho} + mg$.

where $m$ = mass of the body. $R$ = centrif. force.

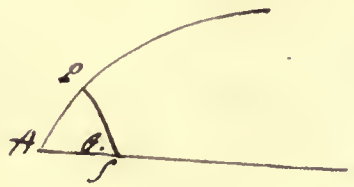
At the lowest point $v^2 = 2ga$, if $a$ = rad of the $\odot$ or the length of the string. also $\rho = r = a$.

$\therefore$ at the lowest point we have

$$T = 2mg + mg = 3mg.$$

= 3 times the weight of the body.

Find the time of describing a given angle of the orbit round the focus.




we have $r^2 d\theta = h$.

$$\therefore \underset{t}{dt} = \frac{r^2}{h} = \frac{m^2}{h} \cdot \sec^4 \frac{\theta}{2}.$$

$$\text{or } \underset{\theta}{dt} = \frac{2m^2}{h} \cdot \left(1 + \tan^2 \frac{\theta}{2}\right) \underset{\theta}{d}\left(\tan \frac{\theta}{2}\right)$$

$$\therefore \underset{x}{dt} = \frac{2m^2}{h} \cdot \left\{ \underset{x}{d}\left(x + x^3\right) \right\} \therefore t = \frac{2m^2}{h} \left(x + \frac{1}{3} x^3\right) + C.$$

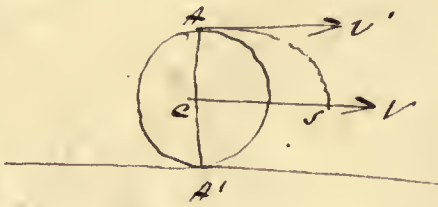
$$\therefore t = \frac{2m^2}{h} \left\{ \tan \frac{\theta}{2} + \frac{1}{3} \tan^3 \frac{\theta}{2} \right\} + C.$$

when $t = 0$ $\theta = 0$ $\therefore C = 0$.

$$t = \frac{2m^2}{h} \left\{ \tan \frac{\theta}{2} + \frac{1}{3} \tan^3 \frac{\theta}{2} \right\}.$$

A molecule is projected from the wheel of a coach in rapid motion. given the velocity of the coach, find the greatest distance before the wheel to which the molecule can be thrown.

CV the direction of the motion of the coach.
$v$ = velocity of the coach.




at A' a molecule on the surface of the wheel would be at rest, at A its velocity would be in direct. $+v'$ // to the horizontal plane, and $= 2v$.
let AS represent the path of the molecule after being projected at A. when its vel. would evidently be a maximum. because the whole angular velocity at that point is in the direction of the motion. of the coach.

let $r$ = rad of the wheel.

Then if $t$ be the time the molecule takes to arrive at S

$$t = \sqrt{\frac{2r}{g}} \quad \therefore \quad CS = 2v \sqrt{\frac{2r}{g}}.$$

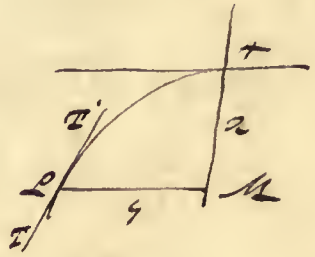
But during this interval the axis of the wheel would have advanced thro' a space. $v\sqrt{\frac{2r}{g}}$.

$\therefore$ the dist. of the particle before the axle of the wheel $= v\sqrt{\frac{2r}{g}}$. and before the point in nearest point in the circum. of the wheel $= v\sqrt{\frac{2r}{g}} - r$.

when the molecule reaches the ground its distance before the axis of the wheel $= 2v\sqrt{\frac{r}{g}}$. which is the req^d. distance.

A body acted on by gravity moves on the convex surface of the cycloid whose vertex is the highest point. the velocity at the highest point being given, det. the point where it will leave the cycloid and the latus rectum of the parabola described

x.y. coords of the reqd. pt P.




Let R = reaction on the curve inwards.

$$R = mg\frac{dy}{ds} - \frac{mv^2}{\rho}.$$

$$\rho = -\frac{1}{d^2y/dx^2}\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}} = \frac{x^{\frac{3}{2}}\sqrt{2a-x}}{a}\cdot\left(\frac{2a}{x}\right)^{\frac{3}{2}} = 2\sqrt{x}.\sqrt{2a-x}$$

$$\therefore \frac{dy}{ds} = \frac{dy/dx}{\sqrt{1+(dy/dx)^2}} = \frac{\sqrt{2a-x}}{\sqrt{2a}}$$

$$\therefore R = mg\left\{\frac{\sqrt{2a-x}}{\sqrt{2a}} - \frac{(h+x)}{\sqrt{2a}\sqrt{2a-x}}\right\}$$ if $h$ = space due to velocity at T.

at the reqd point $R = 0$

$$\therefore \frac{mg}{\sqrt{2a}}\left\{\frac{2a-x-h-x}{\sqrt{2a-x}}\right\} = 0 \quad \text{or} \quad x = \frac{1}{2}(2a-h).$$

Again at L if LT be a tangent. $\tan T'Ly = \frac{1}{dy/dx} = \sqrt{\frac{x}{2a-x}}$

$$= \sqrt{\frac{\frac{1}{2}(2a-h)}{2a-a+\frac{h}{2}}} = \sqrt{\frac{a-\frac{h}{2}}{a+\frac{h}{2}}}.$$

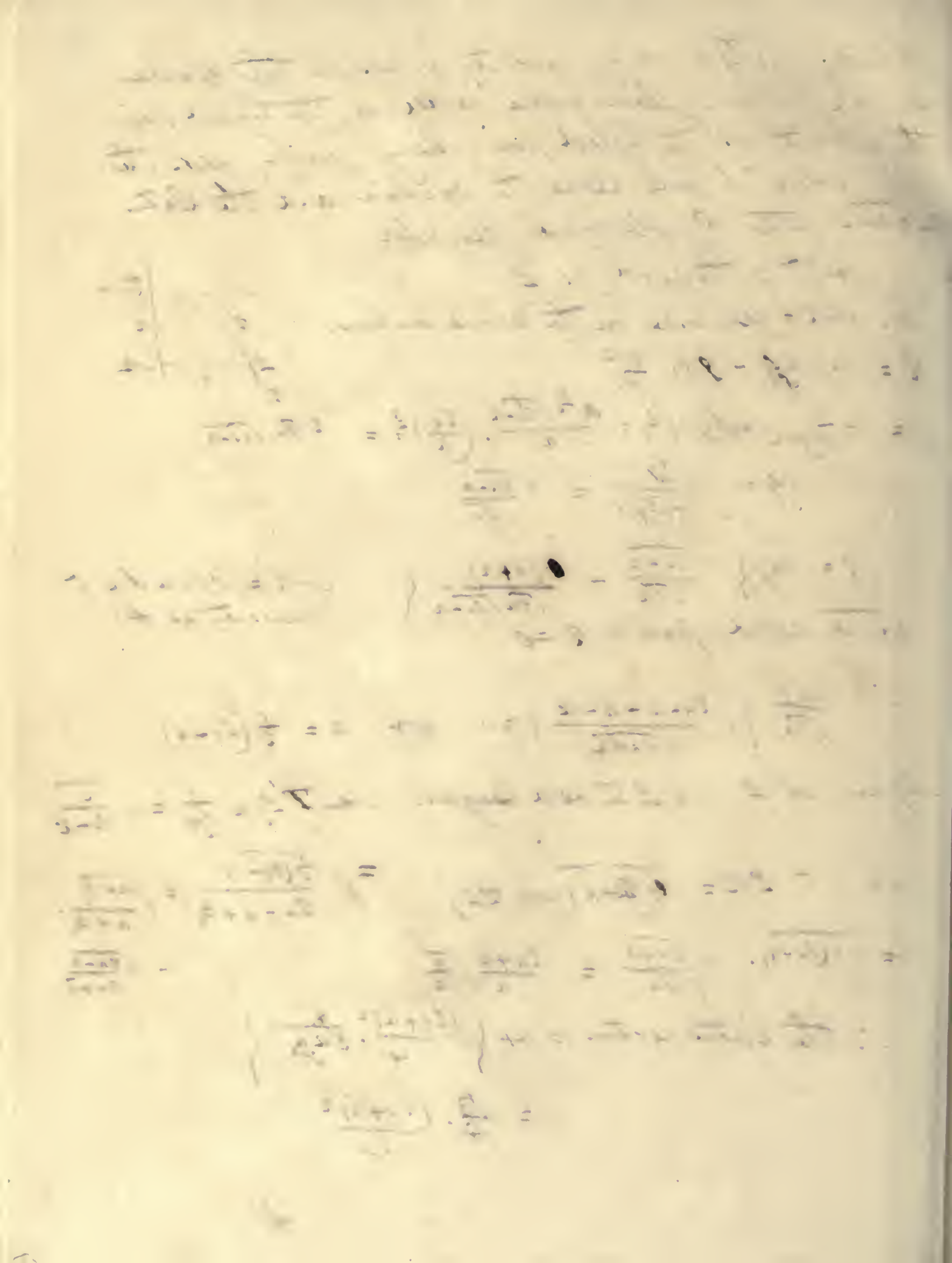
$$= \sqrt{\frac{2a-h}{2a+h}}.$$

$\therefore$ vel // to $LM = \sqrt{g(2a+h)}.\cos TLy$

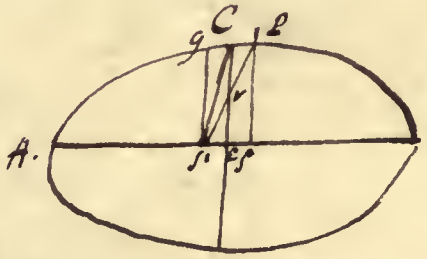
$$= \sqrt{g(2a+h)}.\frac{\sqrt{2a+h}}{\sqrt{4a}} = \frac{2a+h}{2}\sqrt{\frac{g}{a}}.$$

$\therefore$ the latus rectum $= 4\left\{\frac{(2a+h)^2}{4}\cdot\frac{g}{2ag}\right\}$

$$= \frac{2}{g}\cdot\frac{(2a+h)^2}{2a}.$$



A body moves in an ⊙ of small eccentricity and is acted on by a central force tending to the focus; then that its angular distance from the extremity of the nearer apse will be determined nearly by the eqn. $\theta = \frac{\pi}{2} + 2\tan^{-1} e$, at the end of $\frac{1}{4}$ of the periodic time.




S the focus of the ⊙ S'L nearly bisects CC' in V since the ⊙ is small. Hence area CVL = area S'VC' nearly or S.L is nearly the portion of the radius vector at the end of $\frac{1}{4}$ of the periodic time

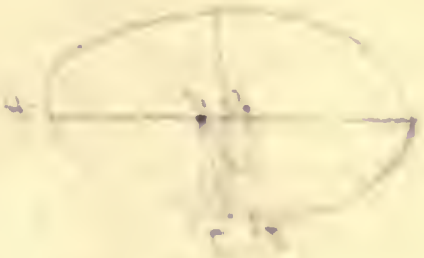
Now $gSA = \frac{\pi}{2}$.

Then $gS'C = \tan CS_1L$ nearly $= \frac{S_1C}{CC_1} = \frac{ae}{b} = \frac{ae}{a\sqrt{1-e^2}}$

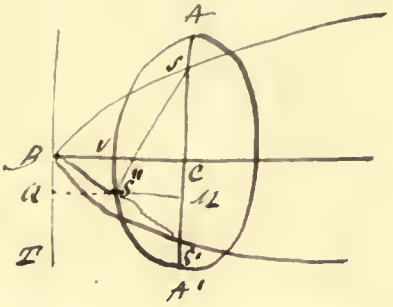
$= \frac{ae}{a} = e$ nearly

$\therefore gS_1L = 2\tan^{-1} e$ nearly.

and the whole $\angle ASL = \frac{\pi}{2} + 2\tan^{-1} e$ nearly.



A body describes a Ω about a centre of force in a point in the circumference of a given Δ the foci of which are in the circumf of the Δ the force varying inversely as (dist)$^2$. Shew that the time of moving from one focus to the other is the same, at whatever point in the circumference of the Δ the centre of force is placed.




$y^2 = 4mx$ the $=^n$ to the Δ

$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$ $=^n$ to the Ω

S'' the centl. of force $CM = x$. $MS = y$ coordts of S

Then the whole area traced out by the radius vector $= \frac{2}{3} SS' \cdot BC - \frac{1}{2} SS' \cdot SM$

$= SS' \left\{ \frac{2}{3} BC - \frac{1}{2} \frac{b}{a} \sqrt{a^2 - x^2} \right\}$.

$= \frac{2}{3} SS' \left\{ BC - \frac{3}{4} \frac{b}{a} \sqrt{a^2 - x^2} \right\}$

Let $v'$ be the velocity at B.

$\therefore h = v \cdot SB \sin SBT + v' = v \cdot SV + v'$

$= v \cdot \left( BC - \frac{b}{a} \sqrt{a^2 - x^2} \right) + v'$

$= v'' \cdot BC - \frac{3}{4} \frac{b}{a} \sqrt{a^2 - x^2} \Big\}$.

$\therefore v' = \frac{v}{4} \frac{b}{a} \sqrt{a^2 - x^2}$

Then $T = \frac{2}{3} \frac{SS'}{v''}$ which is constant.

Given the pressure on the sides of a wedge, the coefficient of fricn. the weight of the wedge. and that of a hammer which strikes it, and the modulus of elasticity between their surfaces. to find how far one blow of the hammer with given velocity will drive the wedge.



$L$ = pressure on the wedge.

$\therefore \mu L$ = friction

$m$ = mass of wedge.

$m'$ = mass of hammer.

$v'v$. velocity of hammer immediately before and immediately after impact

$$m'(v'-v) = mv. \qquad -\mu L$$

For since gravity ~~and friction~~ requires time to act. the weight of the wedge will not alter the initial velocity.

$$m'v' - \mu L = (m+m')v \qquad \therefore v = \frac{m'v' - \mu L}{m+m'}.$$

$\therefore$ for the motn of the wedge.

$$\frac{d^2x}{t} = \frac{\mu L}{m} - g.$$

$$\frac{dx}{t} = \left(\frac{\mu L}{m} - g\right)t + \frac{m'v' - \mu L}{m+m'} \qquad (1)$$

$$x = \frac{1}{2}\left(\frac{\mu L}{m} - g\right)t^2 + \frac{m'v' - \mu L}{m+m'}t.$$

which gives the req$^d$ space $x$ in terms of $t$.

But from (1), if $\frac{dx}{t} = 0$ $\quad t = \dfrac{(m'v' - \mu L/m}{(m+m')(\mu L - mg)}$.

$$\therefore x = \frac{1}{2}\cdot\frac{\mu L - mg}{m}\left\{\frac{(m'v' - \mu L)^2 m}{(m+m')^2(\mu L - mg)^2} + \frac{(m'v' - \mu L)^2 m}{(m+m')^2(\mu L - mg)}\right.$$

$$= \frac{3}{2}\cdot\left(\frac{m'v' - \mu L}{m+m'}\right)^2\left(\frac{m}{\mu L - mg}\right).$$

Camb. Problems.

Let $V$ = Moment of the impulse communicated.

$x$ = distance from the centre of the point of its application and suppose it to act at right angles to that distance.

$a$ = rad of the earth $\therefore \frac{4}{3}\pi a^3$ = its volume.

Now. $V$ acting at $x$ is equivalent to $V$ acting at the earths centre, and a couple whose moment is $Vx$.

If $V$ were applied at the earths centre it would produce a velocity $= \frac{3V}{4\rho\pi a^3}$ in the direction of the impulse communicated.

Let $r$ = distance of the earth from the Sun. at the time the impulse was communicated.

$\therefore P = \frac{2\pi . a'^{\frac{3}{2}}}{\sqrt{\mu}}$ where $a'$ = semi major axis of the earths orbit

But $\frac{3V}{4\rho\pi a^3} = \sqrt{\frac{\mu}{r}\left(\frac{2a'-r}{a'}\right)}$ $\therefore \sqrt{\mu} = \sqrt{\frac{a'r}{2a'-r}} \cdot \frac{3V}{4\rho\pi a^3}$

$\therefore P = \frac{8}{3} \cdot \frac{\pi^2 a' \rho a^3}{V} \sqrt{\frac{2a'-r}{r}}$.

Let $\alpha$ be the angular velocity communicated to the earth by the couple whose moment is $Vx$.

Take an elementary section of the earth $\perp$ to the axis of revolution, at a distance contained between planes at a distance $x$ and $x+\delta x$ from the centre.

The product of each particle of this elementary $\odot$ into its velocity about the axis $= \int_0^{2\pi} \int_x^a r^2 \delta\theta \delta r . r\alpha$

$= \frac{2}{24}\pi\alpha q^4 = \frac{1}{2}\pi\alpha . \left\{a^2 - x^2\right\}^{\frac{2}{2}}$

$\therefore$ for the whole of the mchine. $\frac{2}{5}\pi\alpha \int_x^a \rho\alpha \left(a^2 - x^2\right)^{\frac{3}{2}}$

$\frac{1}{2}\pi\alpha . \int_x^{a-a} \left(a^2 - x^2/2.$

$(a^2+x^2)^2 = a^4 + 2a^2x^2 + x^4.$

$\int_x (a^2+x^2)^2 = a^4x + \frac{2}{3}a^2x^3 + \frac{1}{5}x^5.$

$\int_{-a}^{a} (a^2+x^2)^2 = 2a^5 + \frac{4}{3}a^5 + \frac{2}{5}a^5 = \frac{a^5}{15}\{30+20+6\}$

$= \frac{56}{15}a^5$

∴ the moment of the sphere about its axis $= \frac{28}{15}a^5\pi\rho$

∴ $Fx = \rho\alpha \cdot \frac{28}{15}a^5\pi$ or $\alpha = \frac{15Fx}{28\rho\pi a^5}$

but time of diurnal rotation $= \frac{2\pi}{\alpha}$

$= \frac{56\pi^2\rho a^5}{15Fx}$

suppose in round numbers the earth makes 365 daily rotations, during the period of an annual revolution.

∴ $\frac{365 \times 56\pi^2\rho a^5}{15Fx} = \frac{8}{3} \cdot \frac{\pi^2 a'\rho a^3}{V}\sqrt{\frac{2a'-x'}{x'}}$

$\frac{73 \times 7a^2}{x} = a'\sqrt{\frac{2a'-x'}{x'}}$

∴ $x = \frac{5329a^2x'}{a'(2a'-x')}$

1)

Let $x^2+y^2+z^2 = c^2$ be the $\frac{1}{8}$th of the sphere.

Then $dx\,dy\,dz$ = volume of element.

$\therefore V = \int dx\,dy\,dz$ .

$= \int dx \int dy\, z = \int dx\, zy + C.$

$= \int dx \int dy \cdot \sqrt{c^2-(x^2+y^2)}$ .

$= \int dx \left\{ \frac{y}{2}\sqrt{c^2-x^2-y^2} + \frac{c^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{c^2-x^2}} \right.$ .

The limits of $y$ are $y=0$ and $y = \sqrt{c^2-x^2}$.

$\therefore V = \int_0^c \frac{c^2-x^2}{2} \cdot \frac{\pi}{2}$ .

$= \left(\frac{c^3}{2} - \frac{c^3}{6}\right)\frac{\pi}{2} = \frac{\pi c^3}{6}$ = vol of the eighth part of the sphere.

$\therefore \frac{4}{3}\pi c^3$ = vol of the whole sphere.

p 15

Let $a$ be the depth of the lowest part of the orifice from the surface of the fluid.

Let $x$ be the depth of any horizontal section of the fluid below the surface.

Then the (velocity)$^2$ of the fluid at this depth $= 2gx$.

$\therefore 2y\sqrt{2gx}.\delta x =$ volume of the fluid discharged in a unit of time, contained between horizontal sections of the fluid at distances $x$ and $x+\delta x$ from the surface.

$\therefore \int_x^a . 2y\sqrt{2gx} = 4\sqrt{2gm}.\int_x^a . x = 2\sqrt{2gm}.a^2$. is the quantity of fluid discharged during a unit of time.

also the area of the $\mathcal{P} = \frac{4}{3}xy = \frac{8}{3}.a\sqrt{ma}$.

$\therefore$ the mean velocity $= 3.\frac{2a^2\sqrt{2gm}}{8a\sqrt{ma}} = \frac{3}{4}.\sqrt{2ga}$.

and the horizontal section which has this velocity is situated at a depth $= \frac{9a}{16}$ from the $\mathcal{O}$.

(f 14.)

Let by $\frac{c}{r} = \frac{\theta}{a}$ be the = an Equiangular spiral

Then $v^2 = c - 2\int_r \frac{\mu}{r^2} = 2\mu\left(\frac{1}{r} - \frac{1}{a}\right)$ where = original distance of the particle

$\therefore \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = 2\mu\cdot\left(\frac{1}{r} - \frac{1}{a}\right)$.

but $d\theta = \frac{a}{r}\,dr$. $\therefore r^2\left(\frac{d\theta}{dt}\right)^2 = a^2\left(\frac{dr}{dt}\right)^2$.

$\therefore \left(1+a^2\right)\left(\frac{dr}{dt}\right)^2 = 2\mu\left(\frac{1}{r} - \frac{1}{a}\right)$.

$$\frac{dr}{dt} = \frac{1}{\sqrt{1+a^2}}\cdot\sqrt{\frac{2\mu}{a}}\cdot\sqrt{\frac{a-r}{r}}.$$

$$\frac{dt}{dr} = \sqrt{1+a^2}\sqrt{\frac{a}{2\mu}}\cdot\left\{\frac{\frac{a}{2}-r}{\sqrt{ar-r^2}} - \frac{a}{2}\cdot\frac{1}{\sqrt{ar-r^2}}\right\}.$$

$$\therefore t = \sqrt{\frac{a(1+a^2)}{2\mu}}\cdot\left\{\sqrt{ar-r^2} - \frac{a}{a}\sin^{-1}\frac{2r}{a}\right) + C.$$

at the commencement of motion $t = 0$. $r = 2a$.

$$0 = \sqrt{\frac{a(1+a^2)}{2\mu}}\cdot\left\{-\frac{a}{2}\pi\right) + C.$$

$$\therefore t = \sqrt{\frac{a(1+a^2)}{2\mu}}\cdot\left\{\frac{a}{a}\left(\pi - \sin^{-1}\frac{2r}{a}\right) + \sqrt{ar-r^2}\right).$$

after one revolution $r = ce^{-\left(\frac{\theta}{a}-2\pi\right)}$.

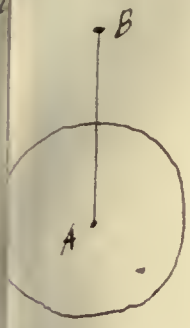
$$\therefore t' = \sqrt{\frac{a(1+a^2)}{2\mu}}\cdot\left\{\frac{a}{2}\left(\pi - \sin^{-1}\frac{2ce}{a}\right) + \sqrt{a\cdot ce^{-\left(\frac{\theta}{a}-2\pi\right)} - c^2e^{-\left(\frac{2\theta}{a}-4\pi\right)}}\right\}$$

when the particle arrives at the ctr. $r = 0$.

$$\therefore t'' = \frac{a^{\frac{3}{2}}\pi}{2^{\frac{3}{2}}}\cdot\sqrt{\frac{1+a^2}{\mu}}\Bigg\}.$$

$$\therefore \frac{t'}{t''} = \frac{\frac{a}{2}\left(\pi - \sin^{-1}\frac{2ce^{-\left(\frac{\theta}{a}-2\pi\right)}}{a}\right\} + 2\sqrt{ace^{-\left(\frac{\theta}{a}-2\pi\right)} - c^2e^{-\left(\frac{2\theta}{a}-4\pi\right)}}}{a\pi}$$

p 164




Let the lowest point in the circular arc be taken for the origin of coord$^{s}$. and let $x.y$ be the coord$^{s}$ of the centre of the sphere at a time $t$.

$\therefore \frac{d^2s}{dt^2} = -g\frac{dx}{ds}$ $\therefore \left(\frac{ds}{dt}\right)^2 = -2gx + c$.

at the commencement of the motion let $x = h$.

$\therefore \frac{ds}{dt} = \sqrt{2g(h-x)}$. or $\frac{dt}{ds} = \frac{1}{\sqrt{2g}\sqrt{h-x}}$.

$\therefore \frac{dt}{dx} = \frac{ds}{\sqrt{2g}\cdot\sqrt{h-x}}$. In the $\odot$ $\frac{ds}{dx} = \frac{a}{\sqrt{2ax-x^2}}$

$\therefore \frac{dt}{dx} = \frac{a}{\sqrt{2g}}\cdot\frac{1}{\sqrt{hx-x^2}}\cdot\frac{1}{\sqrt{2a-x}} = \frac{1}{2}\sqrt{\frac{a}{g}}\cdot\frac{1}{\sqrt{hx-x^2}}\left\{1+\frac{1}{2}\left(\frac{x}{2a}\right)+\frac{1.3}{2.4}\left(\frac{x}{2a}\right)^2+\right.$

$\therefore t = -\frac{1}{2}\sqrt{\frac{a}{g}}\left\{\text{vers}^{-1}\frac{2x}{h} + \frac{1}{4a}\left(\sqrt{hx-x^2}+\frac{h}{2}\text{vers}^{-1}\frac{2x}{h}\right)+\&c.\right\}+C.$

when $t=0$ $x = h$.

$\therefore 0 = \frac{1}{2}\sqrt{\frac{a}{g}}\cdot\left\{\pi + \frac{h}{4a}\pi + \&c\right\}+C.$

$\therefore t = \frac{1}{2}\sqrt{\frac{a}{g}}\cdot\left(\text{vers}^{-1}\frac{2x}{h} + \frac{1}{4a}\left(\sqrt{hx-x^2}+\frac{h}{2}\text{vers}^{-1}\frac{2x}{h}\right)+\&c\right)$

at the time of a complete oscillation $h = -x$.

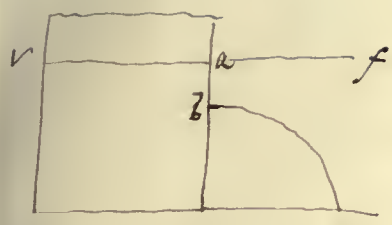
$\therefore T = \sqrt{\frac{a}{g}}\cdot\frac{5\pi}{4}$. $= \frac{\pi}{2}\sqrt{\frac{a}{g}}\left\{1+\frac{h}{8a}\right.$.

$\therefore T = \frac{\pi}{2}\sqrt{\frac{a}{g}}$ nearly.

Let $l$ be the distance from B measured in the direction B A of the centre of oscillation

$\therefore l = \frac{h^2+h^2}{h} = \frac{\frac{2}{5}\cdot\frac{a^2}{4}+a^2}{\frac{4}{5}a} = \frac{22}{20}a = \frac{11}{10}a.$

p131




at the section of the surface of the fluid made by a vertical plane passing thro' the orifice
$ab = c$. $\pi c^2$ = area of section of the cylinder.
Then the velocity of the issuing stream is $\sqrt{2gc}$.
also $af$ is the direction of the $\therefore$ the
Colored section. $= 4c$. $\therefore$ It suffices.

Let $x$ = depth of the surface below the horizontal line $va$
at a time $t$.

$$\therefore \frac{dx}{dt} \cdot \pi b^2 = a\sqrt{2g(c-x)}.$$

$$dt_x = \frac{\pi b^2}{a\sqrt{2g}} \cdot \frac{1}{\sqrt{c-x}}.$$

$$t = -\frac{\pi b^2}{a\sqrt{2g}} \cdot \frac{1}{2}\sqrt{c-x} + C.$$

at the commencement of motion $x = 0$.

$$\therefore t = \frac{\pi b^2}{a\sqrt{2g}} \cdot \frac{1}{2}\left\{\sqrt{c} - \sqrt{c-x}\right).$$

when the fluid reaches $b$ $x = c$.

$\therefore$ if $T$ = the time req'd.

$$T = \frac{\pi b^2}{2a} \cdot \sqrt{\frac{c}{2g}} \cdot \}.$$

Let $l$ be the length of a pendulum which
vibrates once during this period.

$$\therefore T = \pi\sqrt{\frac{l}{g}}.$$

$$\therefore \frac{b^2}{2a}\sqrt{\frac{c}{2}} = \sqrt{l} \cdot \quad \therefore l = \frac{cb^4}{8a^2}.$$

$$\therefore \frac{I_1}{l} = \frac{4c}{\frac{cb^4}{8a^2}} = \frac{32a^2}{b^4}$$

p. 13

h.g.




Let $\alpha$ = inclination of the plane to the horizon. take BA for the axis of $x$.

$\therefore \frac{dv}{dt} = g\sin\alpha - \mu v^2$

$\therefore \frac{dv}{g\sin\alpha - \mu v^2} = dt$ or $dt = \frac{1}{\mu}\cdot\frac{1}{\frac{g}{\mu}\sin\alpha - v^2}$.

$\therefore t = \frac{1}{2\sqrt{\mu g}}\cdot \log\left\{\frac{\sqrt{g\sin\alpha} - \sqrt{\mu}v}{\sqrt{g\sin\alpha} + \sqrt{\mu}v}\right\}$. (1)

Again $v\frac{dv}{dx} = g\sin\alpha - \mu v^2$.

$\frac{v\,dv}{g\sin\alpha - \mu v^2} = dx$.

$-\frac{1}{2\mu}\cdot\frac{d(g\sin\alpha - \mu v^2)}{g\sin\alpha - \mu v^2} = dx$.

$\therefore -\frac{1}{2\mu}\cdot\log.(g\sin\alpha - \mu v^2) = x + c$.

$-\frac{1}{2\mu}\cdot\log(g\sin\alpha) = c$.

$\therefore \frac{1}{2\mu}\cdot\log.\frac{g\sin\alpha}{g\sin\alpha - \mu v^2} = x$.

$e^{2\mu x} = \frac{g\sin\alpha}{g\sin\alpha - \mu v^2}$

$\therefore \mu v^2 e^{2\mu x} = g\sin\alpha(1 - e^{2\mu x})$

$\therefore v = \sqrt{\frac{g\sin\alpha}{\mu}}\cdot\sqrt{e^{-2\mu x} - 1}$

$\therefore t = \frac{1}{2\sqrt{\mu g}}\cdot\log\left\{\frac{1 - \sqrt{e^{-2\mu x} - 1}}{1 + \sqrt{e^{-2\mu x} - 1}}\right\}$.




$$\rho = \frac{1}{a^2 y_{xx}} \left(1 + (y_x)^2\right)^{\frac{3}{2}}.$$

$$y_x = \frac{-x}{\sqrt{a^2 - x^2}} \quad \therefore \quad 1 + (y_x)^2 = \frac{a}{\sqrt{a^2 - x^2}}$$

$$y_{xx} = \frac{-\sqrt{a^2 - x^2} + x^2}{(a^2 - x^2)^{\frac{3}{2}}} = \frac{-a^2}{(a^2 - x^2)^{\frac{3}{2}}}$$

$$\therefore \rho = \frac{(a^2 - x^2)^{\frac{3}{2}}}{a^2} \left\{ \frac{-a^2}{(a^2 - x^2)^{\frac{3}{2}}} \right\} = a$$

$$\therefore X - a = + \frac{y_x}{x^2 y} \left(1 + (y_x)^2\right).$$

for $x^2 + y^2$

for $(x-a)^2 + (y-b)^2 = c^2$.

$$Y - b = - \frac{y_x}{x^2 y} \left(1 + (y_x)^2\right)$$

$$\therefore x - a + (y - b) y_x = 0.$$

$$X - a = \frac{+x}{\sqrt{a^2 - x^2}} \cdot \frac{(a^2 - x^2)^{\frac{3}{2}}}{a^2} \left(\frac{a^2}{(a^2 - x^2)}\right)^{\frac{3}{2}}$$

$$= \frac{x}{a^2} \sqrt{a^2 - x^2}, = x$$

$$\therefore X = x + a.$$

$$Y = \frac{(a^2 - x^2)^{\frac{3}{2}}}{a^2} \cdot \frac{a^2}{\sqrt{a^2 - x^2}}$$

$$= \sqrt{a^2 - x^2}.$$

$$Y - b = \sqrt{a^2 - (X - a)^2} = \sqrt{2aX - X^2}$$

$$Y = b + \sqrt{2aX - X^2}$$

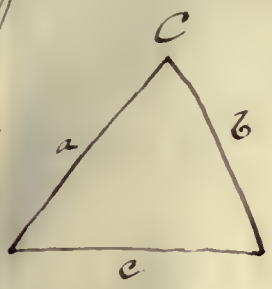
$$d_x^2 z + 3 d_x d_y z + 2 d_y^2 z. = x + y.$$

$$\left(d_x^2 + 2 d_y^2\right)\left(d_x + d_y\right)$$

$$\Delta z = \theta^{m(x+y)}$$

$$\therefore d_x^3 z = \left(m^3 + 3m^2 + 2m^2\right)\theta^{m_1(x+y)} = (x+y)\,\theta$$

$$6m^2 \cdot \theta^{m_1(x+y)} = (x+y).$$




$c$ the base. $C$ the constant vertical angle.

$$\cos C = \frac{a^2+b^2-c^2}{2ab} \quad \therefore 1+\cos C = \frac{(a+b)^2-c^2}{2ab}$$

$$\text{or } 2\cos^2\frac{C}{2} = \frac{(a+b+c)(a+b-c)}{2ab} = \frac{s(s-2c)}{2ab}$$

where $s$ = perimeter

$$\therefore s^2 - 2cs = A \cdot ab$$

$$s^2 - 2cs + c^2 = A \cdot ab + c^2$$

$$s - c = \sqrt{A \cdot ab + c^2}$$

$$s = c + \sqrt{A \cdot ab + c^2}$$

$$ds = \frac{A \cdot (a\,db + b\,da)}{\sqrt{A \cdot ab + c^2}} = 0.$$

$$\therefore a\,db + b\,da = 0$$

$$2ab\cos C = a^2 + b^2 - c^2$$

$$2\cos C(a\,db + b\,da) = 2a\,da + 2b\,db$$

$$\therefore \mu \cdot 2\cos C(a\,db - b\,da) = 2\mu a\,da + 2\mu b\,db$$

$$(2\cos C \cdot a + 2b)\mu - a = 0$$

$$\therefore \mu = \frac{a}{2(a\cos C - b)}$$

$$(2\cos C\, b - 2a)\mu - b = 0$$

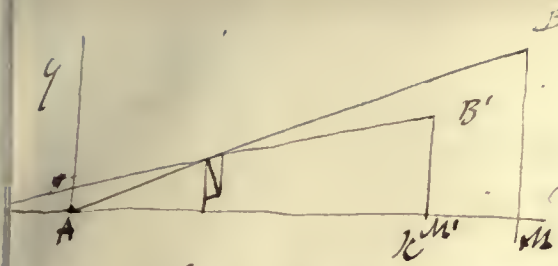
$$\therefore \mu = \frac{b}{2(b\cos C - a)} = \frac{a}{2(a\cos C - b)}$$

which can only be satisfied by $a = b$.

for $ab\cos C - b^2 = ab\cos C - a^2$

$$b^2 = a^2 \text{ or } b = a.$$

(p 5)




Let A B be the initial position of the rod A'B' its position at the expiration of a time t
AM' = x . M'B' = y ; AM = b . MB = c

Then if R = thrust down the rod at time t we have.

$\frac{R}{g} = \text{Area } B'A'M' \quad \therefore \quad R = g \cdot \frac{y}{a}$

∴ the force acting ∥ to AM is equal

$$\frac{d^2x}{t} = -g \cdot \frac{y}{a} \cdot \text{Area } B'A'M' = -\frac{g}{2a} \cdot y\sqrt{a^2-y^2}.$$

$$\frac{d^2y}{t} = g - g\,\frac{y}{a}\,\text{Area } B'A'M' = g - \frac{g}{2a^2}(a^2-y^2)$$

$$\therefore \left(\frac{dy}{t}\right)^2 = -\frac{g}{a^2}\left\{a^2y - \tfrac{1}{3}y^3\right\} + C.$$

at the commencement of motion $\left(\frac{dy}{t}\right)^2 = 0 \cdot y = c$.

$$0 = -\frac{g}{a^2}\left(a^2c - \tfrac{1}{3}c^3\right) + C.$$

$$\therefore \frac{dy}{t} = \sqrt{\frac{g}{a}} \cdot \sqrt{a^2(c-y) - \tfrac{1}{3}(c^3-y^3)}.$$

Now $\frac{dx}{t} = \frac{dx}{y}\frac{dy}{t} = \frac{dx}{y}\sqrt{\frac{g}{a}} \cdot \sqrt{a^2(c-y) - \tfrac{1}{3}(c^3-y^3)}$

$$\therefore \frac{d^2x}{t} = d_t\left\{\frac{dx}{y}\sqrt{\frac{g}{a}}\sqrt{a^2(c-y) - \tfrac{1}{3}(c^3-y^3)}\right\}$$

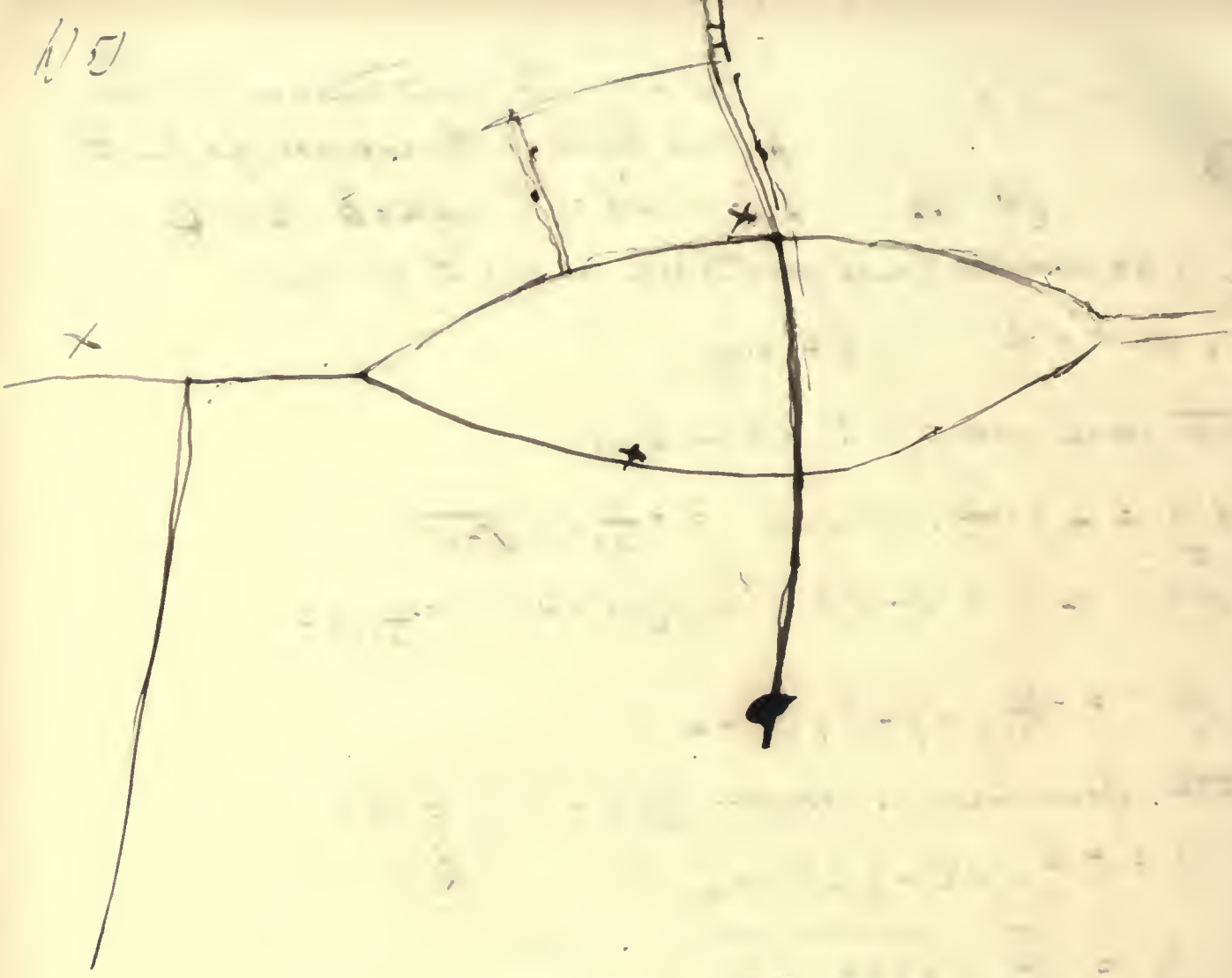
$$= d_y\left\{\frac{dx}{y}\sqrt{\frac{g}{a}}\sqrt{a^2(c-y) - \tfrac{1}{3}(c^3-y^3)}\right\}\left\{d_t y\right\}$$

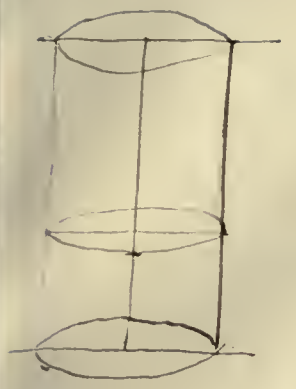
$$= d_y\left\{\frac{dx}{y}\sqrt{a^2(c-y) - \tfrac{1}{3}(c^3-y^3)}\right\} \cdot \left\{\frac{g}{a^2}\sqrt{a^2(c-y) - \tfrac{1}{3}(c^3-y^3)}\right\}.$$

$$\therefore y\sqrt{a^2-y^2} = 2\sqrt{a^2(c-y) - \tfrac{1}{3}(c^3-y^3)}\left\{d_y^2x\sqrt{a^2(c-y) - \tfrac{1}{3}(c^3-y^3)} + dx\,\frac{-a^2+3y^2}{\sqrt{a^2(c-y) - \tfrac{1}{3}(c^3-y^3)}}\right\}.$$

$$= 2d_y^2x\left\{2\left(a^2(c-y) - \tfrac{1}{3}(c^3-y^3)\right\}\right.$$

$$\left. + \frac{(3y^2-a^2)\,dx}{\sqrt{a^2(c-y) - \tfrac{1}{3}(c^3-y^3)}}\right.$$





Let $x$ = depth of the base of the cylinder below the surface of the fluid at a time $t$.

$\therefore \frac{d^2x}{t} = \rho(x - a)$ $\rho$ = density of the fluid. $2a$ = height of the cylinder

$\therefore 2\frac{dx}{t}\frac{d^2x}{t} = 2\rho x\frac{dx}{t} - 2a\rho\cdot\frac{dx}{t}$

$\left(\frac{dx}{t}\right)^2 = \rho(x^2 - 2ax) + C.$

At the commencement of motion $\frac{dx}{t} = 0 \cdot x = 2a.$

$\therefore 0 = \rho \times 0 + C \quad \therefore \quad C = 0$

$\therefore \left(\frac{dx}{t}\right)^2 = \rho(x^2 - 2ax).$

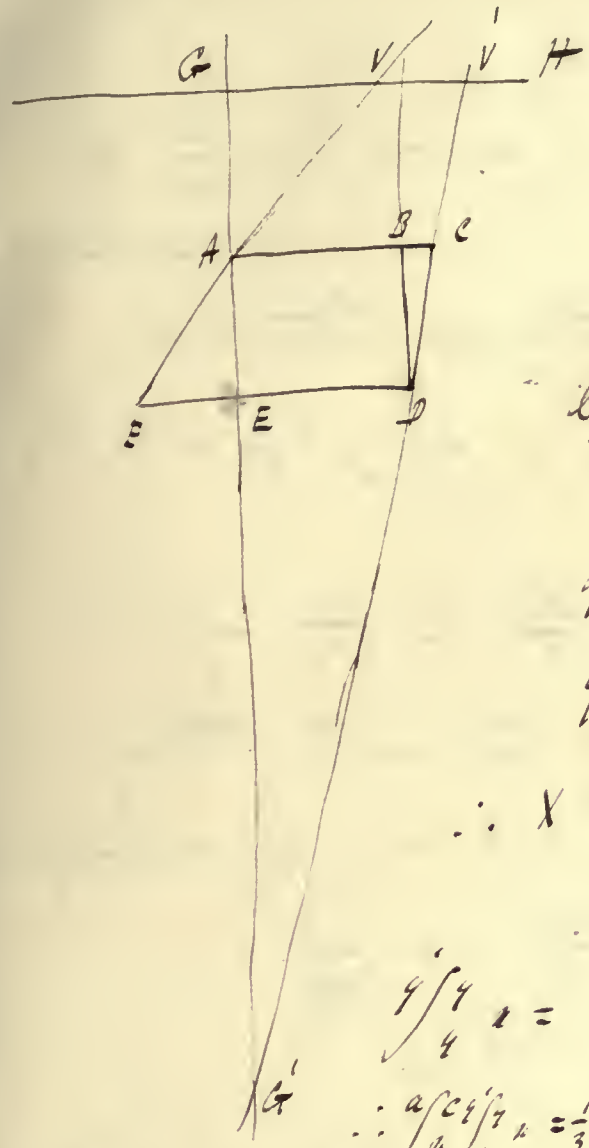
when the cylinder reaches its highest position

$\frac{dx}{t} = 0 \qquad \therefore 0 = x^2 - 2ax$

$\therefore x = 0 \quad$ or $\quad x = 2a.$

wherefore in its highest position the base of the cylinder will be level with the surface of the fluid —

(b. 81)




Take GH for the axis of y.
GE vertical for the axis of x.
Divide the hyperoid into the □ AD whose sides are ∥ to the axis of x & y.
and the Δ's AEE. BCD.

~~let~~ ~~y = mx + b = eq to AE.~~

~~$y = -\frac{b}{a}x + b$ = eq to CD.~~

$$y = -\frac{GH}{GA}x + GV = -\frac{b}{a}x + b = \text{eq to } AE$$

$$y' = -\frac{GV'}{GG'}x + GV' = -\frac{b'}{a'}x + b' = \text{eq to } CD.$$

$$\therefore X = \frac{\int_x^a \int_y^c y\,dy\,dx^2}{\int_x^a \int_y^c dy\,dx} \cdot Y = \frac{\int_x^a \int_y^c y\,dy\,dx}{\int_x^a \int_y^c dy\,dx}$$

$$\int_y^{y'} y\,dy = x(y'-y) = x^2\left(\frac{b}{a} - \frac{b'}{a'}\right) + x(b'-b).$$

$$\therefore \int_x^a \int_y^c y\,dy\,dx = \tfrac{1}{3}(c^3-a^3)\left(\frac{b}{a} - \frac{b'}{a'}\right) + \tfrac{1}{2}(c^2-a^2)(b'-b)$$

Again $\int_y^{y'} y\,dy\,x^2 = x^3\left(\frac{b}{a} - \frac{b'}{a'}\right) + x^2(b'-b)$.

$$\therefore \int_x^a \int_y^{y'} y\,dy\,dx = \frac{1}{4}(c^4-a^4)\left(\frac{b}{a} - \frac{b'}{a'}\right) + \frac{1}{3}(c^3-a^3)(b'-b).$$

$$\int_y^{y'} dy\,xy = \frac{1}{2}\left\{x^3\left(\frac{b'^2}{a'^2} - \frac{b^2}{a^2}\right) + 2x^2\left(\frac{b^2}{a} - \frac{b'^2}{a'}\right) + x(b'^2 - b^2)\right.$$

$$\therefore \int_x^a \int_y^{y'} dy\,xy = \frac{1}{6}(c^4-a^4)\left\{\frac{b'^2}{a'^2} - \frac{b^2}{a^2}\right) + \frac{1}{3}(c^3-a^3)\left(\frac{b^2}{a} - \frac{b'^2}{a'}\right) + \frac{1}{2}(c^2-a^2)(b'^2-b^2)$$

$$\therefore X = \frac{1}{2}\left\{\frac{3(c^4-a^4)\left(\frac{b}{a} - \frac{b'}{a'}\right) + 4(c^3-a^3)(b'-b)}{2(c^3-a^3)\left(\frac{b}{a} - \frac{b'}{a'}\right) + 3(c^2-a^2)(b'-b)}\right\}.$$

$$Y = \frac{1}{2}\left\{\frac{2(c^4-a^4)\left(\frac{b'^2}{a'^2} - \frac{b^2}{a^2}\right) + 4(c^3-a^3)\left(\frac{b^2}{a} - \frac{b'^2}{a'}\right) + 6(c^2-a^2)(b'^2-b^2)}{2(c^3-a^3)\left(\frac{b}{a} - \frac{b'}{a'}\right) + 3(c^2-a^2)(b'-b)}\right.$$

Let $\frac{\mu}{r^2}$ = force acting upon Moon at a distance $r$.

$\therefore$ when the mass of the earth is doubled.

$$F = -\frac{h^2}{2} \cdot d_r\left(\frac{1}{p^2}\right) = \frac{2\mu}{p^2}$$

$$\therefore \frac{h^2}{2}\frac{1}{p^2} = \frac{2\mu}{r} + C$$

at the time the mass of the earth was doubled $p = r = \rho$

$$\therefore \frac{h^2}{2\rho^2} = \frac{2\mu}{\rho} + C$$

$$\therefore C = \frac{h^2\rho - 4\mu\rho^2}{2r^2\rho} = \frac{h^2 - 4\mu\rho}{2\rho^2}$$

$$\therefore \frac{h^2}{2}\frac{1}{p^2} = \frac{2\mu}{r} + \frac{h^2\rho - 4\mu\rho^2}{2\rho^3\rho}$$

$$\frac{r^2\rho h^2}{h^2\rho - 4\mu r^2}\frac{1}{p^2} = 4\mu$$

$$\therefore \frac{h^2}{2}\cdot\frac{1}{p^2} = \frac{2\mu}{r} + \frac{h^2 - 4\mu\rho}{2\rho^2}$$

$$\therefore \frac{\rho^2 h^2}{h^2 - 4\mu\rho}\frac{1}{p^2} = \frac{2\rho}{h^2 - 4\mu\rho}\cdot\frac{2\mu}{r} + 1.$$

$$\therefore \frac{\rho^2 h^2}{4\mu\rho - h^2}\frac{1}{p^2} = \frac{4\mu\rho}{4\mu\rho - h^2}\cdot\frac{1}{r} - 1$$

$$\therefore 2a = \frac{4\mu\rho}{4\mu\rho - h^2} \qquad b^2 = \frac{4\mu\rho - h^2}{\rho^2 h^2}$$

$\therefore$ $\rho$ can be determined

Let $T$. $t$. be the times of oscillations

$\therefore T = \pi\sqrt{\frac{L}{g}}$ $t = \pi\sqrt{\frac{l}{g}}$.

also $nT = (n \pm 1)t$.

$\therefore n^2 L = (n \pm 1)^2 . l$

$\therefore l = \frac{L . n^2}{(n \pm 1)^2}$.

The "n" taken to be referred to its asymptotes whose

$$x^2 - y^2 = a^2. \quad (1)$$

also suppose the force to each || taken axis of x.

$$\therefore \frac{d^2y}{dt^2} = 0 \ (2), \quad \frac{d^2x}{dt^2} = -P \ (3).$$

From (2), $x\frac{dx}{dt} - y\frac{dy}{dt} = 0$. (4)

$$x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 - y\left(\frac{d^2y}{dt^2}\right) - \left(\frac{dy}{dt}\right)^2 = 0.$$

$$\text{or } Px + \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 = 0 \text{ for } \frac{d^2y}{dt^2} = 0.$$

From (4) $\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$

$$\therefore \left(\frac{dy}{dt}\right)^2 = \frac{x^2}{y^2}\cdot\left(\frac{dx}{dt}\right)^2.$$

$$\text{or } \left(\frac{dx}{dt}\right)^2 = \frac{y^2}{x^2}\cdot\left(\frac{dy}{dt}\right)^2.$$

From (2) $\frac{d^2y}{dt^2} = 0 \quad \therefore \frac{dy}{dt} = c$

$$\therefore \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 = y^2\frac{-x^2}{x^2}\left(\frac{dy}{dt}\right)^2$$

$$= \left(y^2 - x^2\right)\frac{c^2}{x^2}.$$

$$\therefore Px + \frac{\left(y^2 - x^2\right)c^2}{x^2} = 0.$$

$$\text{or } Px^3 = a^2c^2.$$

$$P = \frac{a^2c^2}{x^3} \quad \therefore = \frac{\mu}{x^3}.$$

Diff.ti Calculus.

Transform $\iiint V\, dx_1\, dx_2\, dx_3\, dx_4$ to polar coords.

$$x_1 = r \sin\theta \cos\varphi = r\cos\varphi$$

$$x_2 = r \sin\theta . \sin\varphi = r \sin\varphi .$$

$$x_3 = r \cos\theta \cos\psi = u \cos\psi$$

$$x_4 = r \cos\theta \sin\psi = u \sin\psi .$$

$$dx_1\, dx_2 = \left( \frac{dx_1}{r}\frac{dx_2}{\varphi} - \frac{dx_2}{r}\frac{dx_1}{\varphi} \right) dr\, d\varphi$$

$$= r\, dr\, d\varphi .$$

$$dx_3\, dx_4 = \frac{dx_3}{u}\frac{dx_4}{\psi} - \frac{dx_4}{u}\frac{dx_3}{\psi}$$

$$= u\, du\, d\psi .$$

$$\therefore dx_1\, dx_2\, dx_3\, dx_4 = r u .\, du\, dr\, d\varphi\, d\psi .$$

But $u = r\cos\theta$

$$r = r \sin\theta .$$

$$du\, dr = \frac{du}{r}\frac{dr}{\theta} - \frac{du}{\theta}\frac{dr}{r}$$

$$= r\, d\theta\, dr .$$

$$\therefore dx_1\, dx_2\, dx_3\, dx_4 = r u r .\, dr\, d\theta\, d\varphi\, d\psi$$

$$= r^3 \sin\theta \cos\theta .\, dr\, d\theta\, d\varphi\, d\psi .$$

$$\therefore \iiint V .\, dx_1\, dx_2\, dx_3\, dx_4 = \iiint V r^3 \sin\theta \cos\theta .\, dr\, d\theta\, d\varphi\, d\psi .$$

Trace and find the area of the curve

$r = a\cos\theta + b.$

$r^2 = a^2\cos^2\theta + 2ab\cos\theta + b^2.$

$$A = \frac{1}{2}\int_\theta r^2$$

$$= \frac{1}{2}\int_\theta \left\{ \frac{a^2}{2}(1+\cos 2\theta) + 2ab\cos\theta + b^2 \right\}.$$

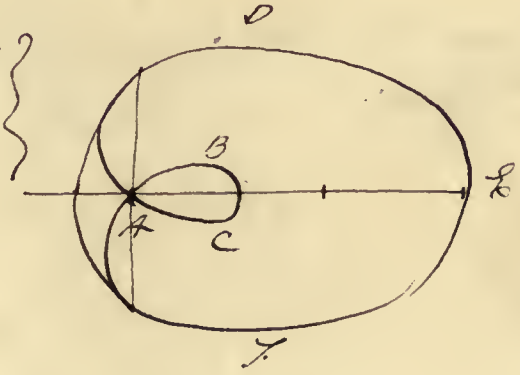
$$= \frac{1}{2}\left\{ \left(\frac{a^2}{2}+b^2\right)\theta + \frac{a^2}{4}\sin 2\theta + 2ab\sin\theta \right\} + C$$

Take double of the value of this integral from $\theta = 0$ to $\theta = \cos^{-1}\frac{b}{a}$.

$$= \frac{1}{2}\left\{ \left(a^2+2b^2\right)\cos^{-1}\frac{b}{a} - \frac{a^2}{2}\cdot\frac{b}{a}\sqrt{1-\frac{b^2}{a^2}} + 4ab\sqrt{1-\frac{b^2}{a^2}} \right\}$$

$$= \frac{1}{2}\left\{ \left(a^2+2b^2\right)\cos^{-1}\frac{b}{a} - b\sqrt{a^2-b^2} + 4b\sqrt{a^2-b^2} \right\}$$

$$= \frac{1}{2}\left\{ \left(a^2+2b^2\right)\cos^{-1}\frac{b}{a} + 3b\sqrt{a^2-b^2} \right\}$$




which = area of ABEFA

area of ABC =

$$\frac{1}{2}\left\{ \left(a^2+2b^2\right)\left(\pi - \cos^{-1}\frac{b}{a}\right) - 3b\sqrt{a^2-b^2} \right\}.$$



Determine whether the curve $4y^2+5xy+x^2+2y+3x+4=0$ has asymptotes, and find the curve when it has.

$$4y^2+5xy+x^2+2y+3x+4=0.$$

$$y^2+\tfrac{1}{4}(2+5x)y+\tfrac{1}{64}\{2+5x\}^2=\tfrac{1}{64}\{2+5x\}^2-\tfrac{1}{64}\{64+48x+16x^2\}$$

$$=\tfrac{1}{64}\{4+20x+25x^2-64-48x-16x^2\}.$$

$$y+\tfrac{1}{8}(2+5x)=\tfrac{1}{8}\{9x^2-28x-60\}^{\frac{1}{2}}.$$

$$=\tfrac{3x}{8}\left\{1-\tfrac{28}{9}\tfrac{1}{x}-\tfrac{60}{9x^2}\right\}^{\frac{1}{2}}.$$

$$=\tfrac{3x}{8}\left\{1-\tfrac{1\cdot 28}{2\cdot 9}\cdot\tfrac{1}{x}+\alpha\right\}.$$

$$=\tfrac{3x}{8}-\tfrac{7}{12}+\alpha.$$

$y=\pm\left\{\tfrac{1}{4}x-\tfrac{29}{24}\right\}$ is the curve of the asymptote.

The lack of the general curve having asymptotes might have been deduced from $b^2-4ac=25-16=9$ a positive integer or from the curve's representing an $\infty$.

Trace the curve whose $= 0$ is $5y^2 + 3xy + 2x^2 + 5y + 6x + 7 = 0$.

Here $b^2 - 4ac = 9 - 40 = -31$.

$$f(hk) = f + \frac{ae^2 - bde + cd^2}{-31} = \frac{5 \cdot 36 - 3 \cdot 6 \cdot 5 + 2 \cdot 25}{-31} + 7 = -\frac{35}{31}.$$

$\therefore$ A and C are of the same sign. and diff. from. $f(hk)$
or the curve is an Ø

$$h = \frac{2ae - bd}{-31} = \frac{2 \cdot 5 \cdot 6 - 3 \cdot 5}{-31} = -\frac{15}{31}$$

$$k = \frac{2ad - be}{-31} = \frac{2 \cdot 2 \cdot 5 - 36}{-31} = \frac{16}{31}.$$

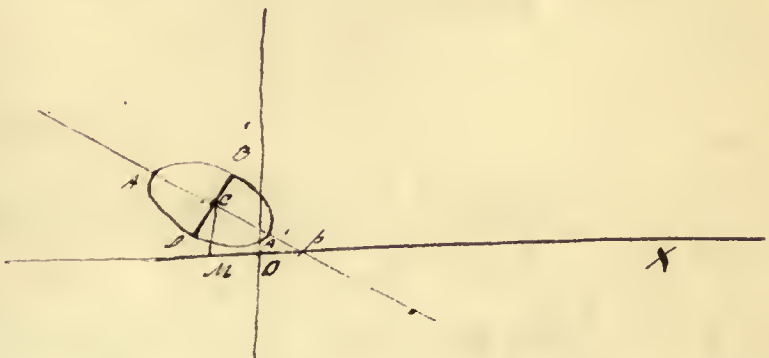
$$\text{Tan } 2\theta = \frac{-b}{a-c} = \frac{-3}{3} = -1 \quad \text{or} \quad \theta = 22\tfrac{1}{2}^\circ.$$

$$A = \frac{1}{2}\left\{a + c + \sqrt{(a-c)^2 + m}\right\} = \frac{1}{2}\left\{7 + \sqrt{65}\right\}.$$

$$B = \frac{1}{2}\left\{a + c - \sqrt{(a-c)^2 + m}\right\} = \frac{1}{2}\left\{7 - \sqrt{65}\right\}.$$

$\therefore Ay^2 + Bx^2 = -f(hk)$ becomes.

$$\frac{31(7 + \sqrt{65})}{714} y^2 + \frac{31(7 - \sqrt{65})}{714} x^2 = 1$$




The curve is represented by the fig. where O is the original, C the new origin of co-ords. $OM = -\frac{15}{31}$, $MC = \frac{16}{31}$. $\angle CpX = 15^\circ 22\tfrac{1}{2}'$.

$$CA' = \left\{\frac{31(7 + \sqrt{65})}{714}\right\}^{-\frac{1}{2}}. \quad CB' = \left\{\frac{714}{31(7 - \sqrt{65})}\right\}^{\frac{1}{2}}. \text{ the semi-axes.}$$

$u = (a+bx^n)^m = y^m$ suppose.

$\therefore \frac{du}{dx} = my^{m-1}\frac{dy}{dx}$.

$= m.(a+bx^n)^{m-1}.\, nbx^{n-1}$

$= mnb.\, x^{n-1}.(a+bx^n)^{m-1}$.

$\frac{d^2u}{dx^2} = mnb\{(n-1)x^{n-2}.(a+bx^n)^{m-1} + (n-1)x^{2n-2}.bm.(a+bx^n)^{m-2}$

$= mnb.\, x^{n-2}.(a+bx^n)^{m-2}\{(n-1)(a+bx^n) + (m-1)bnx^n$.

$= \text{————————}\{a(n-1) - bx^n + mn.bx^n\}$.

$\text{——}\quad\quad\{(n-1)a + (mn-1)bx^n\}$

$\frac{d^n}{dx^n}.(a+bx^n)^m = mnb.\frac{d^{n-1}}{dx^{n-1}}\{x^{n-1}.(a+bx^n)^{m-1}\}$.

$= mnb\{(n-1)(n-2)\text{—}3.2.1.(a+bx^n)^{m-1}$

$+ \{m(n-1)\text{—}3.2.1.(n-1)nbx^n.(a+bx^n)^{m-2}\}$.

$+ \{\frac{n.(n-1)^2}{12}(n-2)\text{—}3.(n-1)(n-2)n^2b^2x^{2n}.(a+bx^n)^{m-3}\}$

$\&c.$ L

$u = \{x + (1+x^2)^{\frac{1}{2}}\}^{\frac{1}{2}} = y^{\frac{1}{2}}$.

$\frac{du}{dx} = \frac{1}{2} y \cdot \frac{dy}{dx} = \frac{1}{2}\{x+(1+x^2)^{\frac{1}{2}}\}^{-\frac{1}{2}} \{1 + \frac{x}{(1+x^2)^{\frac{1}{2}}}$

$= \frac{1}{2} \left\{ \frac{x+(1+x^2)^{\frac{1}{2}}\}^{\frac{1}{2}}}{(1+x^2)^{\frac{1}{2}}} \right.$

$\frac{d^n u}{dx^n} = \frac{1}{2} \frac{d^{n-1}}{dx^{n-1}} \left\{ x+(1+x^2)^{\frac{1}{2}}\right\}^{\frac{3}{2}} \cdot (1+x^2)^{-\frac{1}{2}} \}$.

$= \frac{1}{2} \cdot \frac{d^{n-1}}{dx^{n-1}} \cdot (1+x^2)^{-\frac{1}{2}} \cdot \left(x+(1+x^2)^{\frac{1}{2}}\right]^{\frac{3}{2}} + \alpha$.

$u' = (1+x^2)^{-\frac{1}{2}}$.

$du' = -x \cdot (1+x^2)^{-\frac{3}{2}}$.

$\frac{d^2 u'}{dx^2} = -(1+x^2)^{-\frac{3}{2}} + 3x^2 \cdot (1+x^2)^{-\frac{5}{2}}$.

$= -(1+x^2)^{-\frac{3}{2}} + 3\cdot(1+x^2)^{-\frac{3}{2}} - 3\cdot(1+x^2)^{-\frac{5}{2}}$.

$= 2\cdot(1+x^2)^{-\frac{3}{2}} - 3\cdot(1+x^2)^{-\frac{5}{2}}$.

$\frac{d^3 u'}{dx^3} = -2\cdot 3\cdot x\cdot(1+x^2)^{-\frac{5}{2}} + 3\cdot 5\cdot x\cdot(1+x^2)^{-\frac{7}{2}}$.

$\frac{d^4 u'}{dx^4} = -2\cdot 3\cdot(1+x^2)^{-\frac{5}{2}} + 2\cdot 3\cdot 5\cdot(1+x^2)^{-\frac{5}{2}} - 2\cdot 3\cdot 5\cdot(1+x^2)^{-\frac{7}{2}}$

$+ 3\cdot 5\cdot(1+x^2)^{-\frac{7}{2}} - 3\cdot 5\cdot 7\cdot(1+x^2)^{-\frac{9}{2}} + 3\cdot 5\cdot 7\cdot(1+x^2)^{-\frac{9}{2}}$.

$= 2\cdot 3\cdot 4\cdot(1+x^2)^{-\frac{5}{2}} - 3\cdot 5\cdot 8\cdot(1+x^2)^{-\frac{7}{2}} + 3\cdot 5\cdot 7\cdot(1+x^2)^{-\frac{9}{2}}$.

$\frac{d^5 u'}{dx^5} = -3\cdot 5\cdot x\{2\cdot 4\cdot(1+x^2)^{-\frac{7}{2}} - 8\cdot 7\cdot(1+x^2)^{-\frac{9}{2}} + 7\cdot 9\cdot(1+x^2)^{-\frac{11}{2}}$.

$$u = \{x + (1+x^2)^{\frac{1}{2}}\}^{\frac{1}{2}}.$$

$$\frac{du}{dx} = \frac{1}{2}\{x + (1+x^2)^{\frac{1}{2}}\}^{-\frac{1}{2}} \cdot \left\{1 + \frac{x}{(1+x^2)^{\frac{1}{2}}}\right\} = \frac{1}{2} \cdot \frac{\{x + (1+x^2)^{\frac{1}{2}}\}^{\frac{1}{2}}}{(1+x^2)^{\frac{1}{2}}}.$$

$$\frac{d^2u}{dx^2} = \frac{1}{2}\frac{d}{dx}\{x + (1+x^2)^{\frac{1}{2}}\}^{\frac{1}{2}} \cdot (1+x^2)^{-\frac{1}{2}}.$$

$$= \frac{1}{2} \cdot \{x + (1+x^2)^{\frac{1}{2}}\}^{\frac{1}{2}} \left\{\frac{(1+x^2)^{-\frac{1}{2}} \cdot 2x}{2} - \frac{3x}{(1+x^2)^{\frac{3}{2}}}\right\}$$

$$= \frac{1}{2} \cdot \frac{\{x + (1+x^2)^{\frac{1}{2}}\}^{\frac{1}{2}}}{(1+x^2)^{\frac{3}{2}}} \left\{\frac{(1+x^2)}{2} - 3x\right\}.$$

$$= \frac{1}{2} \cdot \frac{\{x + (1+x^2)^{\frac{1}{2}}\}}{(1+x^2)^{\frac{3}{2}}} \cdot \frac{\{(1+x^2)^{\frac{1}{2}} + x - 7x\}}{2}.$$

$$= \frac{1}{4} \cdot \frac{\{x + (1+x^2)^{\frac{1}{2}}\}^{\frac{3}{2}}}{(1+x^2)^{\frac{3}{2}}} - \frac{7}{4} x \frac{(x + (1+x^2)^{\frac{1}{2}}\}^{\frac{1}{2}}}{(1+x^2)^{\frac{3}{2}}}.$$

$$u = e^{x^n}.$$

$$\frac{du}{dx} = e^{x^n}\left\{ n x^{n-1} \right\}.$$

$$\frac{d^2u}{dx^2} = e^{x^n}\left\{ n.(n-1)x^{n-2} + n^2.x^{2(n-1)} \right\}.$$

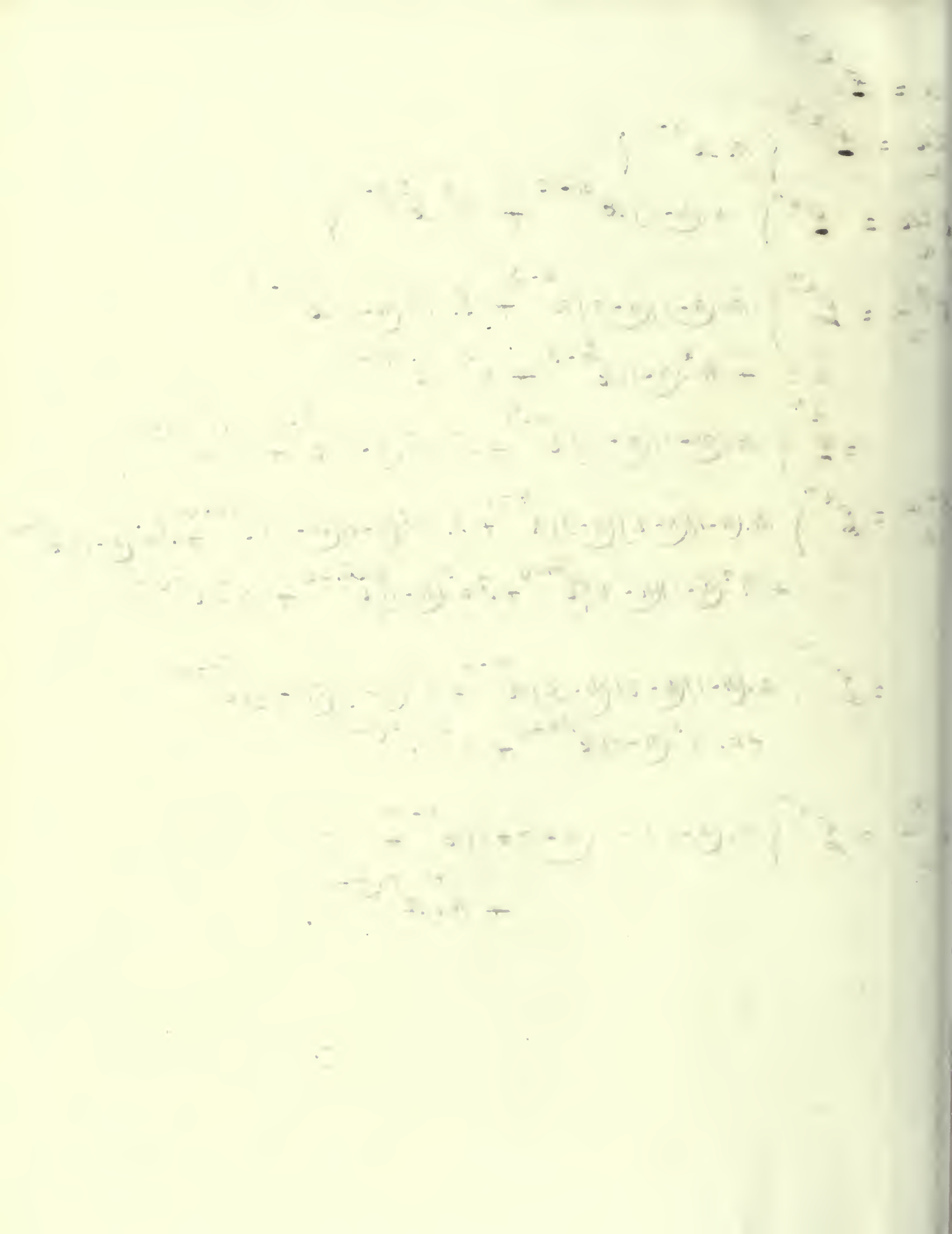
$$\frac{d^3u}{dx^3} = e^{x^n}\left\{ n.(n-1)(n-2)x^{n-3} + 2.n^2.(n-1)x^{2n-3} \right.$$
$$\left. + n^2.(n-1)x^{2n-3} + n^3.x^{3.(n-1)} \right.$$

$$= e^{x^n}\left\{ n.(n-1)(n-2)x^{n-3} + 3.n^2.(n-1)x^{2n-3} + n^3.x^{3(n-1)}. \right\}.$$

$$\frac{d^4u}{dx^4} = e^{x^n}\left\{ n.(n-1)(n-2)(n-3)x^{n-4} + 3.n^2.(n-1)(2n-3)x^{2n-4} + 3n^3.(n-1)x^{3n-4} \right.$$
$$\left. + n^2.(n-1)(n-2)x^{2n-4} + 3n^3.(n-1)x^{2n-4} + n^4.x^{4(n-1)}. \right.$$

$$= e^{x^n}\left\{ n.(n-1)(n-2)(n-3)x^{n-4} + n^2.(n-1)(7n-11)x^{2n-4} \right.$$
$$\left. + 6.n^3.(n-1)x^{3n-4} + n^4.x^{4(n-1)} \right.$$

$$\frac{d^r u}{dx^r} = e^{x^n}\left\{ n.(n-1) - (n-r+1)x^{n-r} + \text{---} \right.$$
$$\left. \text{---} + n^r.x^{r(n-1)} \right.$$



$$u = e^{\sin x}$$

$$\frac{du}{dx} = e^{\sin x} \{ \cos x \}.$$

$$\frac{d^2u}{dx^2} = e^{\sin x} \{ -\sin x + \cos^2 x \} = e^{\sin x} \cdot \{ -\sin x + \tfrac{1}{2}\cos 2x + \tfrac{1}{2} \}.$$

$$\frac{d^3u}{dx^3} = e^{\sin x} \{ -\cos x - \sin 2x - \tfrac{1}{2}\sin 2x + \tfrac{1}{2}\cos 2x \cos x + \tfrac{1}{2}\cos x \}$$

$$= e^{\sin x} \{ -\tfrac{1}{2}\cos x + \tfrac{3}{2}\sin 2x + \tfrac{1}{2}\cos 2x \cos x \}.$$

$$u = \log \{ x + (1+x^2)^{\frac{1}{2}} \}^{-\frac{1}{2}}.$$

$$\frac{du}{dx} = \frac{\left(x + (1+x^2)^{\frac{1}{2}}\right)^{-\frac{1}{2}}}{2(1+x^2)^{-\frac{1}{2}}} \div \left(x + (1+x^2)^{\frac{1}{2}}\right)^{-\frac{1}{2}}.$$

$$= \frac{1}{2} \cdot \frac{1}{(1+x^2)^{\frac{1}{2}}}.$$

$$u = \log(\log x) = \log y \text{ suppose.}$$

$$\therefore \frac{du}{dx} = \frac{dy}{y} = \frac{1}{x \log x}.$$

$$\therefore \int \frac{1}{x} \cdot \log x^{-1} = \int (\log x)^{-1} \cdot d_x \log x = \frac{d_x \cdot \log x^1}{\log x^1} = \log_e \{ \log_e x \}.$$

$$= \frac{\log x}{\log x} - \int x \log x.$$

$$u = \log x^n \quad \therefore = \log . \log x^{n-1} = \log y .$$

$$\therefore \frac{du}{x} = \frac{dy}{y} = \frac{d_x . \log x^{n-1}}{\log x^{n-1}} = \frac{1}{x \log x \log^2 x - \log^{n-1} x} .$$

$$\therefore \int x \frac{1}{x \log x \log^2 x \log^3 x} = \int x . \frac{d_x \log x}{\log x} \frac{1}{\log^2 x \log^3 x} .$$

$$= \int x . \frac{d_x (\log^3 x)}{\log^3 x} = \log . (\log^3 x) = \log^4 (x) .$$

$$u = \log . (\sin x) \quad \therefore \frac{du}{x} = \frac{d_x \sin x}{\sin x} = \cot x .$$

$$\therefore \int x \cot x = \int x \frac{\cos x}{\sin x} = \int x . \frac{d_x \sin x}{\sin x} = \log . (\sin x) .$$

$$u = \log \cos x \quad \therefore \frac{du}{x} = - \frac{\sin x}{\cos x} = - \tan x .$$

$$\therefore \int x \tan x = \int x \frac{\sin x}{\cos x} = - \int x . \frac{d_x \cos x}{\cos x} = - \log . (\cos x) .$$

$$\int \frac{m}{\sin mx} = \int \frac{m}{2 \sin \frac{1}{2} mx \cos \frac{1}{2} mx}$$

$$= \int \frac{m \sec^2 \frac{1}{2} mx}{2 \tan \frac{1}{2} mx} = \int \frac{1}{\tan \frac{1}{2} mx}$$

$$= \log \tan \tfrac{1}{2} mx = \log \left\{ \frac{\sin \frac{1}{2} mx}{\cos \frac{1}{2} mx} \right\}.$$

$$= \log \left( \frac{1 - \cos mx}{1 + \cos mx} \right)^{\frac{1}{2}}.$$

If $u = \log \left( \frac{1 - \cos mx}{1 + \cos mx} \right)^{\frac{1}{2}}$

$$= \tfrac{1}{2} \log (1 - \cos mx) - \tfrac{1}{2} \log (1 + \cos mx)$$

$$= \frac{m}{2} \left\{ \frac{\sin mx}{1 - \cos mx} + \frac{\sin mx}{1 + \cos mx} \right\}.$$

$$= \frac{m \sin mx}{1 - \cos^2 mx} = \frac{m}{\sin mx}.$$

$u = \log(\tan x)$

$$\frac{du}{dx} = \frac{\sec^2 x}{\tan x} = \frac{1}{\cos^2 x}\,\frac{\cos x}{\sin x} = \frac{1}{\sin x \cos x}.$$

$$\therefore \int \frac{1}{\sin x \cos x} = \int \frac{\sec^2 x}{\tan x} = \int \tan x \,\frac{1}{\tan x} = \log(\tan x).$$

$u = \cos(\sin x) \quad = \cos y$

$$\frac{du}{dx} = -\sin y \frac{dy}{dx} = -\sin(\sin x)\cos x.$$

$$\int -\sin(\sin x)\cos x = \int \sin(\sin x)\, d_x \sin x.$$

$$= \cos(\sin x).$$

$u = \sin(\log x).$

$$\frac{du}{dx} = \frac{\cos(\log x)}{x}$$

$$\int_x \cos(\log x)\frac{1}{x} = \int_x \cos(\log x)\, d_x \log x$$

$$= \int_{\log x} \cos(\log x) = \sin(\log x).$$

$$u = \sin^{-1}\frac{x}{(1+x^2)^{\frac{1}{2}}} = \sin^{-1} y.$$

$$\frac{du}{dx} = \frac{\frac{dy}{dx}}{\sqrt{1-y^2}} = + \frac{1}{(1+x^2)^{\frac{3}{2}}} \cdot \frac{1}{\sqrt{1-\frac{x^2}{1+x^2}}}$$

$$= + \frac{1}{1+x^2}$$

$$\int \frac{1}{1+x^2} = \tan^{-1} x = \sin^{-1} \frac{x}{\sqrt{1+x^2}}.$$

(4). $u = \sin^{-1} \frac{1-x^2}{1+x^2}$.

$$\frac{du}{dx} = \frac{-2x(1+x^2) - (1-x^2)2x}{(1+x^2)^2} = \frac{-4x}{(1+x^2)^2}$$

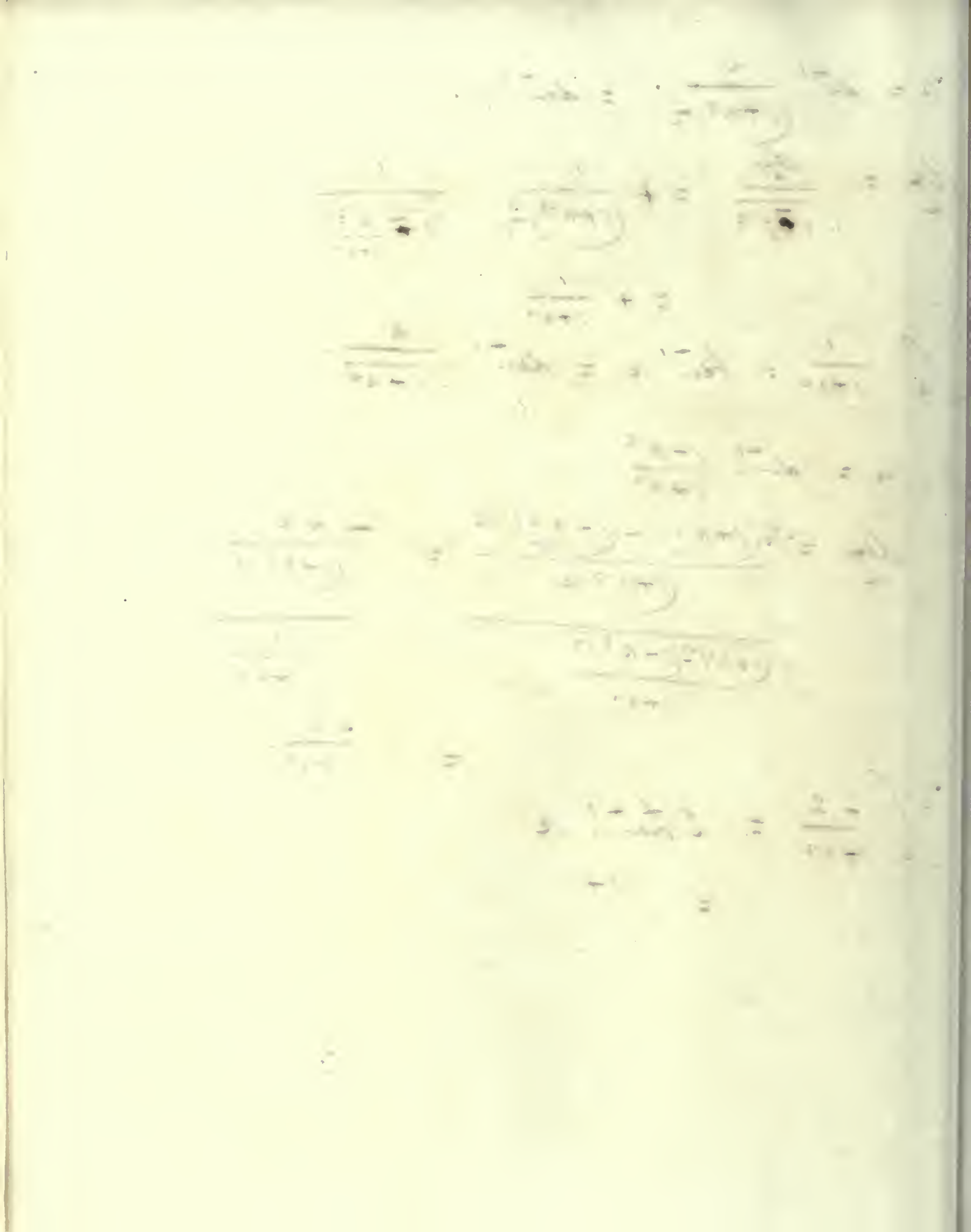
$$\sqrt{\frac{(1+x^2)^2 - (1-x^2)^2}{1+x^2}} \qquad \frac{2x}{1+x^2}$$

$$= \frac{-2}{1+x^2}.$$

$$\therefore \int \frac{-2}{1+x^2} = 2 \tan^{-1} x$$

$$= \qquad 1+$$

$$=$$



$$\therefore d^n w =$$

$$n\left\{(n-1) - (n-r+1).(a+x)^{n-r}.(c+x)^{-n}\right\}\left(1 - \frac{r.n}{n-r+1}\,\frac{a+x}{c+x}\right)$$

$$x^n \log x.$$

$$\frac{d^n w}{dx^n} = n.(n-1).(n-r+1)x^{n-r}.\log x\left\{1 + \frac{r}{n-r+1}\right.$$

$$= n.(n-1) - (n-r+1)x^{n-r}\left\{\log x + \frac{r}{n-r+1}\right.$$

$$\left\{\log x + \frac{r}{n-r+1} - \frac{r.(r-1)}{1.2.}\,\frac{1}{(n-r+1)(n-r+2)}\right.$$

$$w = \Sigma . x^{an+n}.$$

$$\frac{d^r w}{dx^r} = \Sigma\left\{a.x^{ar+n(r-1)} + n.a\,x^{r-1+n-1} + \frac{r(r-1)}{1.2.}\,n.(n-1).a\,x^{r-2+n-2}\right.$$

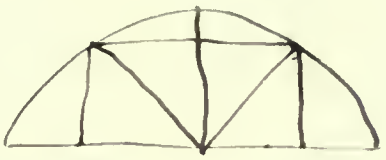
$$\frac{d^n w}{dx^n}.\Sigma a^{an+r} = \Sigma\left\{a x^{n+r} + rn.a\,x^{n-1+r-1} + \frac{r(r-1)}{1.2.}\,n.(n-1)\,a\,x^{r-2+n-2}\right.$$

$$= a.x^{n-r+r-n}$$

$$\therefore \frac{d^n}{dx^n}\Sigma a^{an+r} = a.x^{n-r+r-n}.\frac{d^n}{dx^n}.\Sigma x^{an+n}.$$

$$u = e^{ax} \cdot \cos nx \cdot x^m$$

$$d^r u = (a^2+n^2)^{\frac{r}{2}} \cdot \cos\left(nx + r\frac{\varphi}{2}\right) x^m$$



$$\frac{d^r u}{dx^r} = m(m-1) \cdots (m-r+1) x^{m-r} \cdot e^{ax} \left\{ \cos nx + \frac{r}{1} \cdot \frac{(a^2+n^2)^{\frac{1}{2}}}{m-r+1} \cdot \cos\left(nx+\varphi+\frac{\pi}{2}\right) \right.$$

$$\left. \cos\left(nx+\varphi+\frac{\pi}{2}\right) \right.$$

$$\cos nx + \frac{r}{1} \cdot \frac{(a^2+n^2)^{\frac{1}{2}}}{m-r+1} \cdot \cos\left(nx+\varphi+\frac{\pi}{2}\right) + \frac{r(r-1)}{1\cdot 2} \cdot \frac{a^2+n^2}{(m-r+1)(m-r+2)} \cdots$$

$$\cos nx + \frac{r}{1} \cdot \frac{(a^2+n^2)^{\frac{1}{2}}}{m-r+1} \cdot \cos(nx+\varphi) + \cdots$$

$$= (a^2+n^2)^{\frac{r}{2}} \cdot \cos(nx+r\varphi) x^m$$

$$u = e^{ax} \cdot \cos nx \cdot x^m \cdots$$

$$\frac{d}{dx} u = e^{ax} \cdot (a^2+n^2)^{\frac{r}{2}} \cdot x^m \left\{ \cos(nx+r\varphi) + \frac{r}{x} \cdot \frac{m}{(a^2+n^2)^{\frac{1}{2}}} \cos \frac{\cdots}{x} \right.$$

$$\cdots \frac{d}{dx} \cdot e^{ax} \cdot \cos nx \cdot \sin x$$

$$= e^{ax} \cdot (a^2+n^2)^{\frac{r}{2}} \left\{ \cos(nx+r\varphi) \cdot \sin x + \frac{r}{\sqrt{a^2+n^2}} \cos\{nx+(r-1)\varphi\} \cos x \right.$$

$$\left. - \frac{r(r-1)}{1\cdot 2(a^2+n^2)} \cdot \cos\{nx+(r-2)\varphi\} \sin x \right.$$

$u = \sin nx$.

$\frac{du}{dx} = n \cos nx = n \sin\left(nx + \frac{\pi}{2}\right)$.

$\frac{d^2u}{dx^2} = n^2 \cdot \cos\left(nx + \frac{\pi}{2}\right) = n^2 \sin\left(nx + \pi\right)$.

$\therefore \frac{d^r u}{dx^r} = n^r \cdot \sin\left(nx + \frac{r\pi}{2}\right)$.

$u = \cos nx$.

$\frac{du}{dx} = -n \sin(nx) = n \cos\left(nx + \frac{\pi}{2}\right)$

$\frac{d^2u}{dx^2} = -n^2 \sin\left(nx + \frac{\pi}{2}\right) = n^2 \cos\left(nx + \pi\right)$.

$\therefore \frac{d^r u}{dx^r} = n^r \cdot \cos\left(nx + \frac{r\pi}{2}\right)$.

21).

$$u = \frac{x}{a^2+x^2} = -\frac{1}{2a\sqrt{-1}}\left\{\frac{x}{x+a\sqrt{-1}} - \frac{x}{x-a\sqrt{-1}}\right\}.$$

$$= -\frac{1}{2a\sqrt{-1}}\left\{1 - \frac{a\sqrt{-1}}{x+a\sqrt{-1}} - 1 - \frac{a\sqrt{-1}}{x-a\sqrt{-1}}\right\}.$$

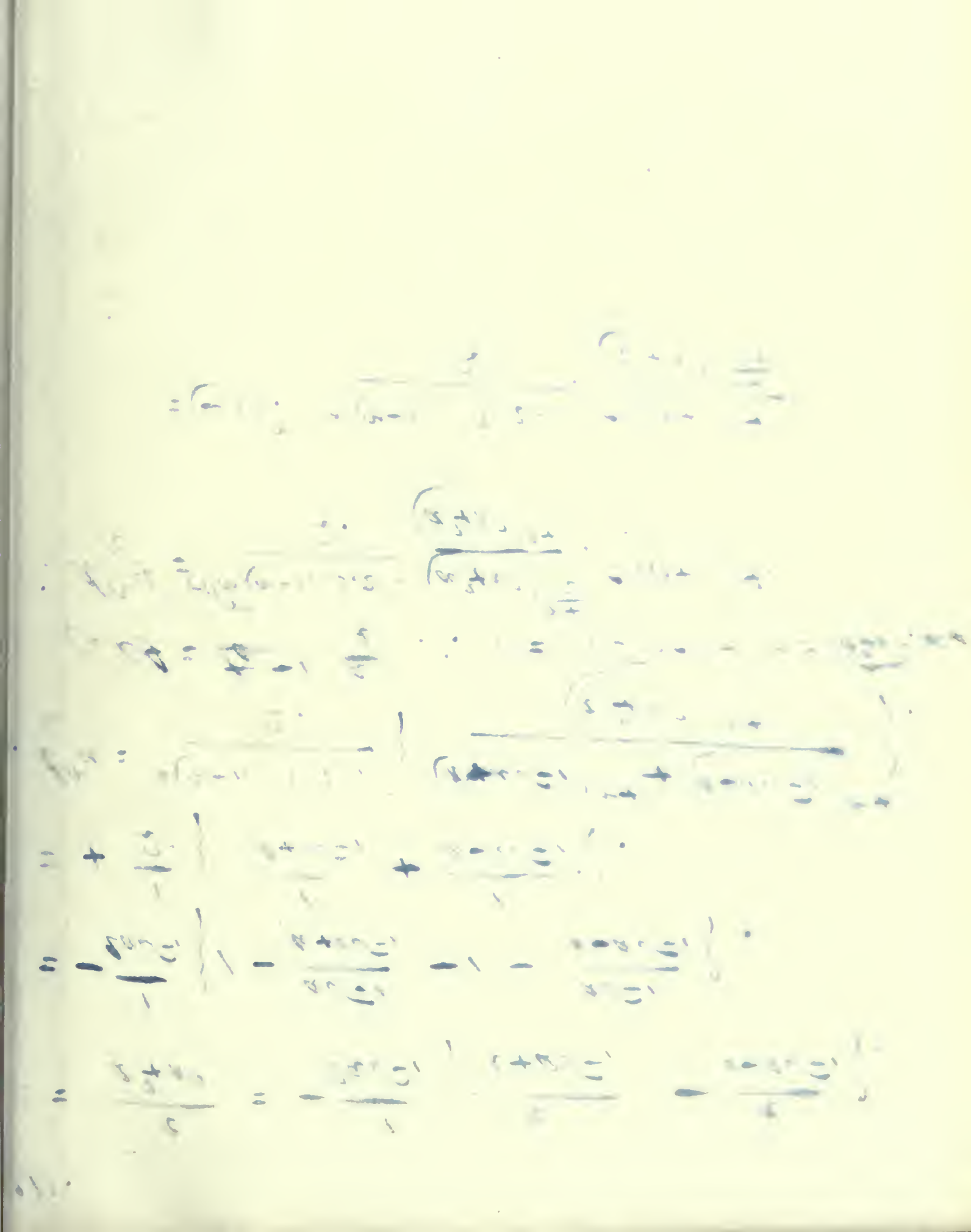
$$= +\frac{1}{2}\cdot\left\{\frac{1}{x+a\sqrt{-1}} + \frac{1}{x-a\sqrt{-1}}\right\}.$$

$$\therefore \frac{d^n u}{dx^n} = \frac{n(n-1\ldots 3\cdot 2\cdot 1}{2}\left\{\frac{(x+a\sqrt{-1})^{n+1} + (x-a\sqrt{-1})^{n+1}}{(a^2+x^2)^{n+1}}\right\}.$$

Let $\theta = \tan^{-1}\frac{a}{x}$ $\therefore x = \sqrt{a^2+x^2}\cos\theta$. $a = \sqrt{a^2+x^2}\sin\theta$.

$$\therefore \frac{d^n u}{dx^n} = \frac{(-1)^n n(n-1\ldots 3\cdot 2\cdot 1}{2}\cdot\frac{(a^2+x^2)^{\frac{n+1}{2}}}{(a^2+x^2)^{n+1}}\cdot\cos(n+1)\theta.$$

$$= (-1)^n\cdot n\cdot\frac{(n-1\ldots 3\cdot 2\cdot 1}{2}\cdot\frac{\cos(n+1)\theta}{(a^2+x^2)^{\frac{n+1}{2}}}.$$



$$u = \frac{x}{a^2+x^2} = -\frac{1}{2a\sqrt{-1}}\left\{\frac{x}{x+a\sqrt{-1}} - \frac{x}{x-a\sqrt{-1}}\right\}.$$

$$\frac{d}{dx}u = -\frac{1}{2a\sqrt{-1}}\left\{\frac{x+a\sqrt{-1}}{x+a\sqrt{-1}}\cdot\frac{-a\sqrt{-1}}{x+a\sqrt{-1}} - \frac{x-a\sqrt{-1}}{x-a\sqrt{-1}}\cdot\frac{-a\sqrt{-1}}{x-a\sqrt{-1}}\right\}.$$

$$= -\frac{1}{2a\sqrt{-1}}\left\{1 - \frac{a\sqrt{-1}}{x+a\sqrt{-1}} - 1 - \frac{a\sqrt{-1}}{x-a\sqrt{-1}}\right\}.$$

$$= +\frac{1}{2}\cdot\left(\frac{1}{x+a\sqrt{-1}} - \frac{1}{x-a\sqrt{-1}}\right).$$

$$\therefore \frac{d^n u}{dx^n} = \frac{(-1)^{n+1}\cdot n(n-1)\ldots 3\cdot 2\cdot 1\sin(n+1)\theta}{(a^2+x^2)^{\frac{n+1}{2}}}.$$

$$\frac{d^n}{dx^n}\cdot\frac{1}{a^2+x^2} = (-1)^n\cdot\frac{n(n-1)\ldots 3\cdot 2\cdot 1\sin(n+1)\theta}{a\cdot(a^2+x^2)^{\frac{n+1}{2}}}.$$

$$\frac{d^n}{dx^n}\cdot\frac{x}{a^2+x^2} = (-1)^n\cdot\frac{n(n-1)\ldots 3\cdot 2\cdot 1\cos(n+1)\theta}{(a^2+x^2)^{\frac{n+1}{2}}}.$$

$$u^n = \frac{1}{(a^2+x^2)} = -\frac{1}{2a\sqrt{-1}}\left\{\frac{1}{x+a\sqrt{-1}} - \frac{1}{x-a\sqrt{-1}}\right\}.$$

$$u^n = \text{[illegible]}$$

$$= (-1)^{r+1}\cdot\frac{r.(r-1).3.2.1}{2a\sqrt{-1}}\left\{\frac{1}{(x+a\sqrt{-1})^{r+1}} - \frac{1}{(x-a\sqrt{-1})^{r+1}}\right.$$

$$= (-1)^{r+1}\cdot\frac{r(r-1).3.2.1}{2a\sqrt{-1}}\left\{.\frac{(x-a\sqrt{-1})^{r+1} - (x+a\sqrt{-1})^{r+1}}{(x^2+a^2.)^{\frac{r+1}{2}}}\right.$$

$$\text{Also } \tan\theta = \frac{a}{x}.$$

$$\therefore \cos\theta = \frac{1}{\sqrt{1+\frac{a^2}{x^2}}} = \frac{x}{\sqrt{a^2+x^2}}$$

$$\therefore x = \cos\theta\sqrt{a^2+x^2}.$$

$$a = \sin\theta\sqrt{a^2+x^2}.$$

$$\therefore (x-a\sqrt{-1})^{r+1} = (a^2+x^2)^{\frac{r+1}{2}}\{\cos(r+1)\theta - \sqrt{-1}\sin(r+1)\theta\}.$$

$$(x+a\sqrt{-1})^{r+1} = (a^2+x^2)^{\frac{r+1}{2}}.\{\cos(r+1)\theta + \sqrt{-1}\sin.(r+1)\theta\}.$$

$$\therefore (x-a\sqrt{-1})^{r+1} - (x+a\sqrt{-1})^{r+1} =$$

$$-2\sqrt{-1}(a^2+x^2)^{\frac{r+1}{2}}.\sin(r+1).\theta.$$

$$\therefore \frac{d^r u^n}{dx^r} = \frac{r.(r-1).3.2.1}{a(a^2+x^2)^{\frac{r+1}{2}}}(-1)^r.\sin.(r+1)\theta.$$

$$u^n = \left(a^2+x^2\right)^{-n}.$$

$$\therefore d_x u^n = n(n+1) - (n+r-1)\left(a^2+x^2\right)^{n+r}(2x)^{-r}\Big\{.$$

$$1 + \frac{n(n+1)\,(2x)^2}{n+r-1\quad (a^2+x^2)}$$

$$= n.(n+1) - (n+r-1).\left(a^2+x^2\right)^{n+r}.(2x)^{-r}.$$

$$-\frac{1}{2a\sqrt{-1}}\cdot\left\{\frac{1}{x+a\sqrt{-1}} - \frac{1}{x-a\sqrt{-1}}\right.$$

$$= \frac{x-a\sqrt{-1}-x-a\sqrt{-1}}{x^2+a^2}$$

$$\therefore \frac{1}{x^2+a^2} = -\frac{1}{2a\sqrt{-1}}\left\{\frac{1}{x+a\sqrt{-1}} - \frac{1}{x-a\sqrt{-1}}\right\}.$$

$$\frac{1}{x+a\sqrt{-1}} = \left(x+a\sqrt{-1}\right)^{-1}$$

$$d_x = -\left(x+a\sqrt{-1}\right)^{-2}.$$

$$d_x^2 = 1.2.\left(x+a\sqrt{-1}\right)^{-3}.$$

$$\therefore d_x^n = n(n-1).3.2.1.\left(x+a\sqrt{-1}\right)^{-n}.$$

$$\therefore u = \frac{1}{x^2+a^2} = -\frac{1}{2a\sqrt{-1}}\left\{\frac{1}{x+a\sqrt{-1}} - \frac{1}{x-a\sqrt{-1}}\right\}.$$

$$\therefore d^n u = (-1)^{n+1}\frac{n(n-1)\,3\cdot 2\cdot 1}{2a\sqrt{-1}}\left\{\frac{1}{(x+a\sqrt{-1})^{n+1}} - \frac{1}{(x-a\sqrt{-1})^{n+1}}\right\}$$

$$= (-1)^{n+1}\cdot\frac{n(n-1)\cdot 3\cdot 2\cdot 1}{2a\sqrt{-1}}\left\{\frac{(x-a\sqrt{-1})^{n+1} - (x+a\sqrt{-1})^{n+1}}{(x^2+a^2)^{\frac{n+1}{2}}}\right\}$$

$$\frac{a}{x} = \tan\theta.$$

$$\therefore \cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{x}{\sqrt{a^2+x^2}}.$$

$$\therefore x = \sqrt{a^2+x^2}\cdot\cos\theta.$$

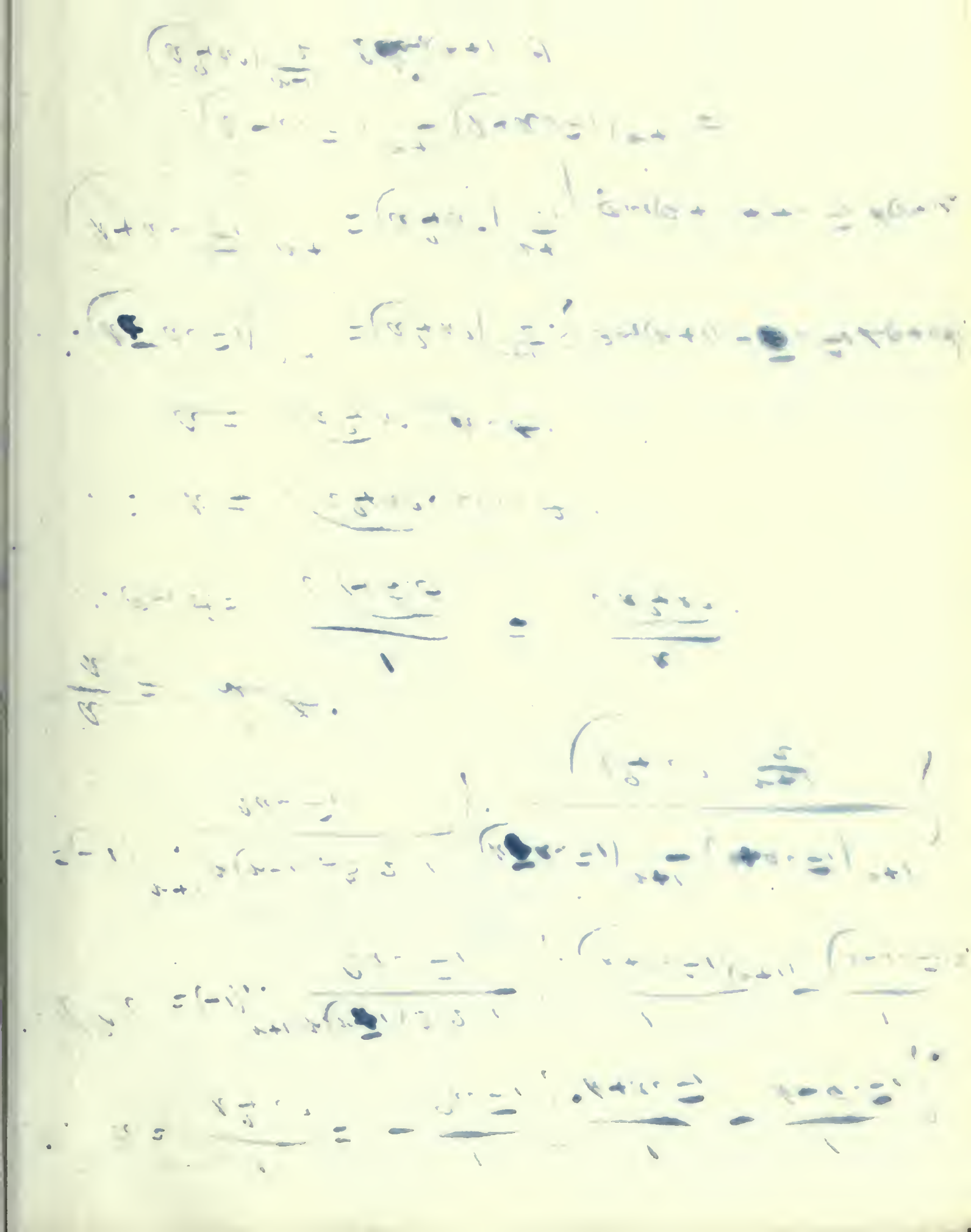
$$a = \sqrt{a^2+x^2}\cdot\sin\theta.$$

$$\therefore (x-a\sqrt{-1})^{n+1} = (a^2+x^2)^{\frac{n+1}{2}}\cdot\{\cos(n+1)\theta - \sqrt{-1}\sin(n+1)\theta\}.$$

$$(x+a\sqrt{-1})^{n+1} = (a^2+x^2)^{\frac{n+1}{2}}\{\cos(n+1)\theta + \sqrt{-1}\sin(n+1)\theta\}.$$

$$\therefore (x-a\sqrt{-1})^{n+1} - (x+a\sqrt{-1})^{n+1} =$$

$$(a^2+x^2)^{\frac{n+1}{2}}\,2\sin(n+1)\theta$$



$u^n = (a^2+x^2)^n$. Here $u = a^2+x^2$.

$u' = 2x$. $c = 1$.

$\therefore x^r u^n = n(n-1)\cdots(n-r+1)\left\{1 + \frac{r(r-1)}{n-r+1}\right.$

$= n(n-1)\cdots(n-r+1)(a^2+x^2)^{n-r}\cdot(2x)^r$

If $n = r$ we have

$\therefore x^r u^n = n(n-1)\cdots 3\cdot 2\cdot 1 \cdot x^n \cdot 2^n$

$\therefore x^r u^n = n(n-1)\cdots(n-r+1)\left\{1 + \frac{r(r-1)}{n-r+1}\cdot\frac{cu}{u'^2} + \frac{r(r-1)(r-2)(r-3)}{(n-r+1)(n-r+2)}\cdot\frac{c^2u^2}{u'^4}\right.$

$\therefore x^r (a^2+4x+9x^2)^n$.

Here $u' = 4+18x$.

$c = 9$.

$n(n-1)(n-2)\cdots(n-r+1)\left\{1 + \frac{r(r-1)}{n-r+1}\cdot\frac{(4+18x)^2(a^2+4x+9x^2)9}{(4+18x)^2}\right.$

If $n = r$ we have

$n(n-1)\cdots 3\cdot 2\cdot 1\left\{1 + \frac{n(n-1)}{1}\cdot\frac{(a^2+4x+9x^2)9}{(2+9x)^2}\right.$

$f(x) = (a+bx+cx^2)^n$.

$f(x+h) = f(x) + f'(x).h + c.h^2)^n$.

$= (u + u'h + ch^2)^n$.

$= (u+u'h)^n + n.(u+u'h)^{n-1}.ch^2 + \frac{n(n-1)}{1\cdot 2}(u+u'h)^{n-2}.c^2h^4$

$\frac{n-1}{1}.u^{n-2}uc.h^3$.

$n-5+3$.

$n-2.u^{n-3}.uc^2.h^5$

$\therefore$ coeff. of $\frac{h^r}{\lfloor r} = \frac{n(n-1)\cdots(n-r+1)}{\lfloor r}.u^{n-r}.u'^r$.

$+\frac{n.(n-1)-(n-r+2)}{1.\lfloor r-2}.u^{n-r+1}.u^{r-2}.c.h^r$

$+\frac{n.(n-1)}{1\cdot 2}.\frac{(n-2)(n-3)\cdots(n-r+3)}{\lfloor r-4}u^{n-r+2}u^{r-4}.c^2.h^r$.

$\therefore$ coeff. of $\frac{h^r}{\lfloor r} =$

$n(n-1)-(n-r+1)u^{n-r}.u^r\left\{1+\frac{(r-1)r}{n-r+1}.\frac{uc}{u'^2}+\frac{(r-3)(r-2)(r-1)r}{1\cdot 2(n-r+1)(n-r+2)}\frac{uc^2}{u'^4}\right.$

$\left(1+\frac{u'}{2u}h\right)^2 = 1+\frac{u'}{u}h+\frac{u'^2}{4u^2}h^2$.

$\frac{d^n u}{h} = n(n-1)-(n-r+1).u^{n-r}.u^r\left\{1+\frac{n.(r-1)}{n-r+1}\frac{cu}{u'^2}.\right.$

$+\frac{r(r-1)(r-2)(r-3)}{(n-r+1)(n-r+2)\cdot 1\cdot 2}.\frac{c^2u^2}{u'^4}$.

$\therefore n = \sqrt{a^2+n^2}.\sin\varphi, \quad a = \sqrt{a^2+n^2}.\cos\varphi.$

$\therefore \frac{du}{dx} = e^{ax}.\sqrt{a^2+n^2}\{\cos\varphi\cos nx - \sin\varphi\sin nx\}.$

$\therefore \frac{du}{dx} = \sqrt{a^2+n^2}\{e^{ax}.\cos(nx+\varphi)\}.$

$\frac{d^2u}{dx^2} = a^2+n^2.\{e^{ax}.\cos.(nx+2\varphi)\}.$

$\frac{d^r u}{dx^r} = (a^2+n^2)^{\frac{r}{2}}.\{e^{ax}.\cos.(nx+r\varphi)\}.$

$u = \log x.$

$\frac{du}{dx} = \frac{1}{x} = x^{-1}.$

$\frac{d^2u}{dx^2} = -x^{-2}. \quad \frac{d^3u}{dx^3} = 2.x^{-3}.$

$\therefore \frac{d^r u}{dx^r} = (-1)^{r-1}.(r-1)\,x^{-r}.$

$$u = e^{x\cos\theta}\cos(x\sin\theta).$$

$$\frac{du}{dx} = e^{x\cos\theta}\{\cos(x\sin\theta)\cos\theta - \sin(x\sin\theta)\sin\theta\}.$$

$$= e^{x\cos\theta}\cdot\cos(\theta + x\sin\theta).$$

$$\frac{d^2u}{dx^2} = e^{x\cos\theta}\{\cos(\theta + x\sin\theta)\cos\theta - \sin(\theta + x\sin\theta)\sin\theta\}.$$

$$= e^{x\cos\theta}\cdot\cos(2\theta + x\sin\theta).$$

$$\therefore \frac{d^n u}{dx^n} = e^{x\cos\theta}\cdot\cos(n\theta + x\sin\theta).$$

$$u = e^{ax}\cos nx.$$

$$\frac{du}{dx} = e^{ax}\{-n\sin nx + a\cos nx\}$$

$$= e^{ax}\cdot\{a\cos nx - n\sin nx\}.$$

Let $\frac{n}{a} = \tan\varphi \quad \therefore \sin\varphi = \frac{\tan\varphi}{\sqrt{1+\tan^2\varphi}} = \frac{n}{\sqrt{a^2+n^2}}$

$$\cos\varphi = \frac{1}{\sqrt{1+\tan^2\varphi}} = \frac{a}{\sqrt{a^2+n^2}}.$$



12.) $u = \frac{1+x}{1-x} = \frac{1-x}{1-x} + \frac{2x}{1-x} = 1 + \frac{2x}{1-x}$.

$$\frac{du}{dx} = \frac{(1-x)2 + 2x}{(1-x)^2} = \frac{2}{(1-x)^2}.$$

$$\therefore \frac{d^2u}{dx^2} = +\frac{2.2}{(1-x)^3} = 2.2.(1-x)^{-3}.$$

$$\frac{d^3u}{dx^3} = 3.2.2.(1-x)^{-4}.$$

$$\therefore \frac{d^nu}{dx^n} = n.(n-1).3.2.2.(1-x)^{-(n+1)}$$

---

$$\frac{d^n(uv)}{dx^n} = v.\frac{d^nu}{dx^n} + \frac{n}{1}\frac{du}{dx}.\frac{d^{n-1}v}{dx^{n-1}} + \frac{n.(n-1)}{1.2}\frac{d^2u}{dx^2}.\frac{d^{n-2}v}{dx^{n-2}}.$$

$$\frac{d^n(uv)}{dx^n} = v.\frac{d^nu}{dx^n} + n.\frac{du}{dx}.\frac{d^{n-1}v}{dx^{n-1}} + \frac{n.(n-1)}{1.2}\frac{d^2u}{dx^2}.\frac{d^{n-2}v}{dx^{n-2}} + \frac{n.(n-1)}{1.2}\frac{d^2v}{dx^2}.\frac{d^{n-3}u}{dx^{n-3}}.$$

13). $u = x^n \cdot (1-x)^n$.

$$d^r(u) = n.(n-1)\cdots(n-r+1)x^{n-r} \cdot (1-x)^n.$$

$$- n^2.(n-1) - (n-r+2)x^{n-r+1} \cdot r.(1-x)^{n-1}$$

$$+ n^2.(n-1)^2 \cdot (n-r+3)^{n-r+2} \cdot \frac{r.(r-1)}{1.2} (1-x)^{n-2}.$$

d. d. — —

$x^n$.

$\frac{du}{dx} = nx^{n-1}$.

$\frac{d^2u}{dx} = n(n-1)x^{n-2}$.

$\frac{d^3u}{dx} = n.(n-1)(n-2)x^{n-3}$.

$\frac{d^r u}{dx} = n.(n-1)(n-2)\ldots(n-r+1)x^{n-r}$.

---

$u = (a+bx)^n$.

$\frac{du}{dx} = n.(a+bx)^{n-1}.b$.

$= nb(a+bx)^{n-1}$

$\frac{d^2u}{dx} = n.(n-1)(a+bx)^{n-2}.b^2$

$= n.(n-1)b^2.(a+bx)^{n-2}$.

$\therefore \frac{d^r u}{dx} = n.(n-1)\ldots(n-r+1)(a+bx)^{n-r}.b^r$

$$u = \frac{1}{x^n} = x^{-n}.$$

$$\therefore \frac{du}{dx} = -n.x^{-(n+1)}$$

$$\frac{d^2u}{dx^2} = n.(n+1)x^{-(n+2)}.$$

$$\frac{d^3u}{dx^3} = -n.(n+1)(n+2)x^{-(n+3)}.$$

$$\therefore \frac{d^ru}{dx^r} = (-1)^r.\ n.(n+1)(n+2)\ldots(n+r-1).x^{-(n+r)}.$$

$$u = a^x.$$

$$\frac{du}{dx} = a^x.\log_e a.$$

$$\frac{d^2u}{dx^2} = a^x.(\log_e a)^2.$$

$$\therefore \frac{d^ru}{dx^r} = a^x.(\log_e a)^r.$$

$$u = e^{mx}.$$

$$\frac{du}{dx} = m.e^{mx} \qquad \frac{d^2u}{dx^2} = m^2.e^{mx}.$$

$$\therefore \frac{d^ru}{dx^r} = m^r.e^{mx}$$

Find the equations to the asymptotes of the curve $y^4+2axy^2-x^4=0$ and trace the curve.

$$y^4+2axy^2-x^4=0.$$

$$y^4+2axy^2+a^2x^2 = x^4+a^2x^2 = x^2(x^2+a^2)$$

$$y^2+ax = x\sqrt{1+\frac{a^2}{x^2}} = x\left(1+\frac{a^2}{2x^2}\right) \text{ if } x=\infty$$

$$\therefore y = \pm\sqrt{x^2-\frac{ax}{2}+\frac{a^2}{2}} \qquad = \pm\left(x^2+\frac{a^2}{2}\right)$$

$$= \pm\left(x-\frac{a}{4}\right). \text{ the equations of a pair of asymptotes.}$$

Also $x=0$ gives $y=0$ and the origin is a point on the curve.

$$(4y^3+4axy)\frac{dy}{dx} = 4x^3-2ay^2 \text{ or } \frac{dy}{dx} = \frac{2x^3-ay^2}{2y^3+2axy} = \frac{0}{0} \text{ at origin}$$

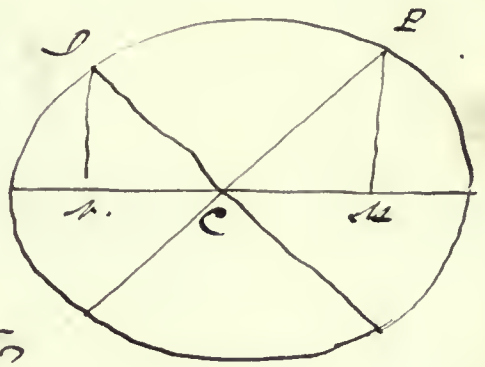
$$p = \frac{d_x\{2x^3-ay^2\}}{dx(2y^3+2axy)} = \frac{6x^2-2ayp}{6y^2p+2ay+2axp} = \frac{0}{0}$$

$$= \frac{12x-2ap^2}{12p^2y+4ap} = -\frac{p^2}{2p} = -\frac{p}{2}.$$

$$\therefore p = p_2 \text{ or } p=0, p=\infty$$

The curve at the origin entitle axes of the asymptotes are

Find those conjugate diameters of an ⊙ whose sum is a max, and whose sum is a min.




Let CE. CD be semi conjugate diameters.

$x, y$ coords of $E$ $\therefore \frac{ay}{b}, \frac{bx}{a}$ are coords of $D$.

$\therefore u = CE + CD = \sqrt{x^2+y^2} + \frac{1}{ab}\sqrt{a^4y^2+b^4x^2}$

$= \sqrt{b^2+e^2x^2} + \sqrt{a^2-e^2x^2}.$

$$\frac{du}{x} = e^2 x \left\{ \frac{1}{\sqrt{b^2+e^2x^2}} - \frac{1}{\sqrt{a^2-e^2x^2}} \right\} = 0.$$

$\therefore e^2x^2 = 0$ and $\frac{1}{\sqrt{b^2+e^2x^2}} - \frac{1}{\sqrt{a^2-e^2x^2}} = 0.$

The former gives $x=0$ and $u = CE+CD = a+b$ a max.

The latter gives $b^2+e^2x^2 = a^2-e^2x^2.$

$$2e^2x^2 = a^2-b^2.$$

$$\left(\frac{a^2-b^2}{a^2}\right)\frac{x^2}{a^2} = \frac{a^2-b^2}{2}.$$

$$x = \frac{a}{\sqrt{2}}. \quad y^2 = \frac{b^2}{a^2}\left(a^2 - \frac{a^2}{2}\right) = \frac{b^2}{2}.$$

$\therefore CE = CD = \frac{\sqrt{a^2+b^2}}{\sqrt{2}}.$ or the conjugate diameters are equal to each other for a minimum.

Prove that at the points of greatest and least curv. the contact of the ⊙ of curv. is of the 3rd order.

we have $\rho = -\frac{1}{d_x^2 y} \cdot \left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}$.

$$\therefore \frac{d\rho}{dx} = -\left\{\frac{(d_x^2 y)^2 \cdot 3 \cdot \left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}}\left(\frac{dy}{dx}\right) - \left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}} \cdot d_x^3 y}{\left(d_x^2 y\right)^2}\right\}$$

$\therefore$ for a max or min we have.

$$3 d_x y \cdot (d_x^2 y)^2 = \left\{1+\left(\frac{dy}{dx}\right)^2\right\} d_x^3 y.$$

$$\text{or } d_x^3 y = \frac{3 \cdot \left(\frac{dy}{dx}\right) \cdot (d_x^2 y)^2}{1+\left(\frac{dy}{dx}\right)^2}.$$

let $(X-b)^2 + (Y-c)^2 = c^2$ be the eq$^n$ to the ⊙ of curv.

$$(X-b) d_x X + Y - c = 0.$$

$$(X-b) d_x^2 X + (d_x X)^2 + d_x Y = 0. \quad \therefore X-b = -\frac{1+(d_x Y)^2}{d_x^2 Y}$$

$$(X-b) d_x^3 Y + 3 d_x X \cdot d_x^2 Y = 0.$$

$$\therefore d_x^3 Y = -\frac{3 d_x X \cdot d_x^2 Y}{X-b} = \frac{3 d_x Y (d_x^2 Y)^2}{1+(d_x Y)^2}.$$

For a contact of the 3rd order. $d_x X = d_x y \cdot d_x^2 Y = d_x^2 y$.

$$d_x^3 Y = d_x^3 y.$$

$$\therefore d_x^3 y = \frac{3 \cdot d_x y \cdot (d_x^2 y)^2}{1+(d_x y)^2}, \text{ which is also}$$

the condit$^n$ of maximum or minimum curvature.

Shewthat a conic $\int^n$ may be dr^d. which shall have a contact of the 4^th. order with a curve at any point; and of at any point of the a P, at the next preceding and succeeding points shall be an $\odot$ and $\mathcal{H}$.

Let $y^2 + 6x^2 + cxy + dy + ex + f = 0$ be the = $^n$ of the $\int^n$. since this contains 5 arbitrary constants, we may assign 5 them such values that at a proposed point $X = y$; $d_x X = d_x y$, $d_x^2 X = d_x^2 y$, $d_x^3 X = d_x^3 y$. $d_x^4 X = d_x^4 y$; $X = f(X)$ being the = $^n$. of the proposed curve $\therefore$ the $\int^n$ may have a contact of the 4^th. order with the curve.

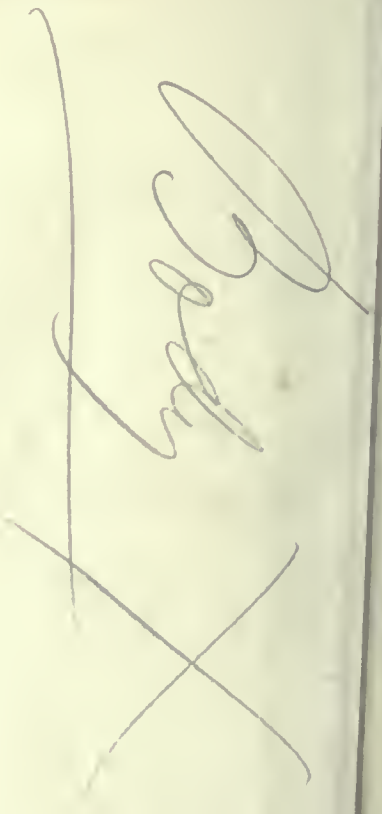
$$\underline{b \; a \; c}$$

$$b^2 - 4ac = 0$$

$$\frac{b}{a} \cdot \frac{b}{c} - 4 = a$$

$d_x^2 y \; d_x^3 y$





Prove that all Ellipses whose axes vary subject to the condition
$a^2 + mb^2 = c^2$ may be touched by 4 straight lines.

Let the $=^n$ Ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{and} \quad a^2 + mb^2 = c^2.$$

Then $\frac{x^2}{a^3} + \frac{y^2}{b^3}\frac{db}{a} = 0 \quad \& \quad a + m\frac{db}{a} = 0$

$$\frac{b^3}{a^3} \cdot \frac{x^2}{y^2} = \frac{db}{a} = \frac{a}{mb}.$$

$$mb^4x^2 = a^4y^2.$$

$$\frac{b^2}{a^2} = \pm \frac{y}{x}\frac{1}{\sqrt{m}}.$$

$$\therefore a^2\left(1 \pm \frac{b^2}{a^2}m\right) = c^2.$$

$$a^2\left(1 \mp \frac{y}{x}\sqrt{m}\right) = c^2.$$

$$c^{-1} = \left(x \pm y\sqrt{m}\right)^2.$$

$$a \pm y\sqrt{m} = \pm c.$$

The $=^n$ 4 straight lines

~~$u = 1 \cdot 3 \cdot 5 \ldots 2n$~~

Let $x = \cos 2z$

$\therefore 1 - x = 2\sin^2 z \qquad 1 + x = 2\cos^2 z$

$\therefore u = 2 \cdot (1 \cdot 3 \cdot 5 \ldots \overline{2n-1}) \left\{ (\sin 2)^{2n-1} \cdot (\cos 2) - \frac{\overline{2n-1} \cdot \overline{2n-2}}{1 \cdot 2 \cdot 3} \cdot (\sin 2)^{2n-3} (\cos 2)^3 + \right.$

$= \frac{1}{n} \cdot \overline{2n-1}! \left\{ 2n \cdot (\sin 2)^{2n-1} \cos 2 - \frac{2n \cdot 2n-1 \cdot 2n-2}{3} \sin 2^{2n-3} \cos 2^3 + \right.$

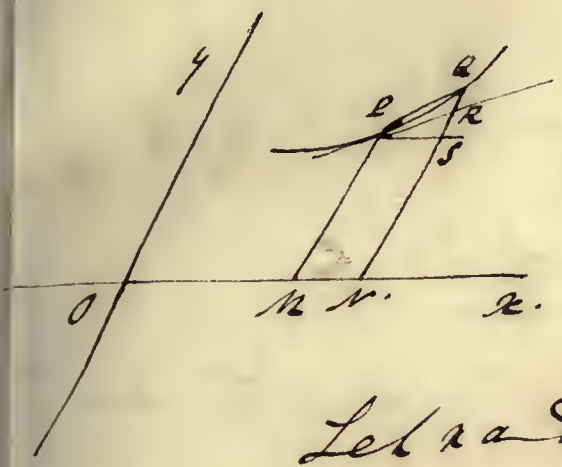
$= \frac{\frac{1}{n} \overline{2n-1}!}{2\sqrt{-1}} \left\{ (\sqrt{-1} \sin 2 + \cos 2)^{2n} - (\sqrt{-1} \sin 2 - \cos 2)^{2n} \right\}$

$= \frac{1}{n} \cdot \frac{\overline{2n-1}!}{2\sqrt{-1}} \left\{ (\cos 2n2 + \sqrt{-1} \sin 2n2) \right.$
$\left. - (\cos 2n2 - \sqrt{-1} \cos 2n2) \right\}$

$= \frac{1}{n} \cdot 1 \cdot 3 \cdot 5 \ldots 2n-1 \sin 2n2$

$= 1 \cdot 3 \cdot 5 \ldots \overline{2n-1} \cdot \frac{\sin(n \cos^{-1} 2)}{n}$

Find the eqn to the tangent at any point of a curve, the axes being oblique.




Let x & y be coords of P.

$MN = h$.

$\therefore x+h$ and $y + \underset{x}{dy}.h + \underset{x}{d^2y}.\frac{h^2}{12}$ = the coords of R.

Then whether the axes be rectangular or oblique.

the eqn to QR is

$$Y - y = \frac{SR}{PS}.\overline{X-x} \qquad (PS \text{ is } \parallel \text{ to } ON).$$

$$Y - y = \frac{\underset{x}{dy}.h + \underset{x}{d^2y}.\frac{h^2}{12}}{h}.\overline{X-x}.$$

$$= \left(\underset{x}{dy}. + \underset{x}{d^2y}.\frac{h^2}{12} + \&c\right)\overline{X-x}$$

$$= \underset{x}{dy}.\overline{X-x} \quad \text{if } h \text{ is indefinitely diminished}$$

Find the value of $\frac{x^3\cos^2 x + \sin x}{x}$ when $x = 0$.

$$u = \frac{x^3\cos^2 x + \sin x}{x} = \frac{0}{0} \text{ if } x = 0.$$

$$= \frac{3x^2\cos^2 x - 2x^3\cos x\sin x + \cos x}{1} = 1 \text{ if } x = 0.$$

Find the value of $\frac{\sqrt{x} - \sqrt{a} + \sqrt{x-a}}{\sqrt{x^2-a^2}}$ when $x = a$.

Let $x = a+h$. $\therefore \sqrt{x^2-a^2} = \sqrt{2ah+h^2} = \sqrt{2ah} + \frac{1}{2}\cdot\frac{h^2}{\sqrt{2ah}} + \alpha$.

$$\sqrt{x-a} = \sqrt{h}.$$

$$\sqrt{x} = \sqrt{a+h} = \sqrt{a} + \frac{1}{2}\frac{h}{\sqrt{a}} + \text{—}$$

$\therefore \frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}$ becomes $\frac{\sqrt{a} + \frac{1}{2}\frac{h}{\sqrt{a}} - \sqrt{a} + \sqrt{h}}{\sqrt{2ah} + \frac{1}{2}\frac{h^2}{\sqrt{2ah}}}$

$$= \frac{\sqrt{h}\left\{1 + \frac{1}{2}\sqrt{\frac{h}{a}} + \alpha\right\}}{\sqrt{h}\left(\sqrt{2a} + \frac{1}{2}\frac{h}{\sqrt{2a}} + \text{—}\right)}$$

$$= \frac{1 + \frac{1}{2}\sqrt{\frac{h}{a}} + \alpha}{\sqrt{2a} + \frac{1}{2}\frac{h}{\sqrt{2a}}} = \frac{1}{\sqrt{2a}} \text{ if } h = 0.$$

Given the bases of 2 Δ's and the sum of their altitudes
perimeters, then that in order that their area may be a
max, the Δ's must be isosceles; and the sides of the Δ's
of the bases must be as the bases.

Let $p, p'$ be the perimeters.
$b, b'$ the bases.
$x, y$; $x', y'$ the other sides of the Δ's
$c$ = sum of the areas.

$$\therefore c = \sqrt{\tfrac{p}{2}\left(\tfrac{p}{2}-b\right)\left(\tfrac{p}{2}-x\right)\left(\tfrac{p}{2}-y\right)} + \sqrt{\tfrac{p'}{2}\left(\tfrac{p'}{2}-b'\right)\left(\tfrac{p'}{2}-x'\right)\left(\tfrac{p'}{2}-y'\right)}. \quad (1),$$

$$p+p' = b+b'+x+y+x'+y' \text{ ——————} \quad (2),$$

From the first $= a$.

$$-\tfrac{1}{2}\sqrt{\tfrac{p}{2}\left(\tfrac{p}{2}-b\right)}\cdot\frac{dx+dy}{\sqrt{\left(\tfrac{p}{2}-x\right)\left(\tfrac{p}{2}-y\right)}} - \tfrac{1}{2}\sqrt{\tfrac{p'}{2}\left(\tfrac{p'}{2}-b'\right)}\cdot\frac{dx'+dy'}{\sqrt{\left(\tfrac{p'}{2}-x'\right)\left(\tfrac{p'}{2}-y'\right)}} = 0.$$

From the 2nd $= a$

$$dx+dy+dx'+dy' = 0.$$
$$\lambda\,dx+\lambda\,dy+\lambda\,dx'+\lambda\,dy' = 0.$$

Collect the coeff$^{t}$ of $dx.dy.\&c.$

$$\left\{1-\tfrac{1}{2}\sqrt{\tfrac{p}{2}\left(\tfrac{p}{2}-b\right)}\right\}\frac{dx}{\sqrt{\left(\tfrac{p}{2}-x\right)\left(\tfrac{p}{2}-y\right)}} = 0 = \left(1-\tfrac{1}{2}\sqrt{\tfrac{p}{2}\left(\tfrac{p}{2}-b\right)}\right)\frac{dy}{\sqrt{\left(\tfrac{p}{2}-x\right)\left(\tfrac{p}{2}-y\right)}}.$$

$$\therefore dy = dx \quad \text{or} \quad x = y.$$

similarly $x' = y'$ and the Δ's are isosceles.

$a^3 y = x^6$.

$a^3 d_x y = 4x^3$.

$a^3 d_x^2 y = 12x^2$

$a^3 d_x^3 y = 24x$.

$a^3 . d_x^4 y = 24$.

∴ Upon taking successive derivatives when $x=0$ no value is obtained ∴ there is no pt. of inflexion at the origin

$y = x^2 \pm x^{\frac{5}{2}}$.

$\frac{dy}{x} = 2x + \frac{5}{2} x^{\frac{3}{2}}$. $= 0$ if $x = 0$.

$\frac{d^2y}{x^2} = 2 + \frac{5}{2} x^{\frac{1}{2}}$. $= 2$ if $x = 0$.

also if $x = 0 - h$ $y = h^2 \pm (-h)^{\frac{5}{2}}$. which is impossible

∴ since $d^2y = 2$. when $x = 0 + h$ or $0 - h$. and $y$ is impossible. there is a cusp of the 2nd. kind at the origin



Three quantities $\overline{a+z}$, $\overline{a+z+h}$, $\overline{a+z+h'}$, approximations to the root $(a)$ of $fax = n$ being substituted for the unknown quantity give results $h$. $h+\delta$, $h+\delta'$; when that $z$ will be very nearly found from the $= n$.

$$z^2(h\delta' - h'\delta) + z(h^2\delta' - h'^2\delta) + hh'(h'-h) = 0.$$

---

let $f(a) = 0$ be the proposed $= n$.

$$\therefore f\overline{a+z} = f(a) + f'(a)z + f''(a)\frac{z^2}{1\cdot 2} + \&c = h.$$

But $f(a) = 0$ for $a$ is a root of the $= n$.

$$\therefore f''(a)\frac{}{1\cdot 2} = h - \frac{f'(a)z}{z^2} \text{ very nearly.}$$

$$f(a+z+h) = f(a) + f'(a)(z+h) + f''(a)\frac{(z+h)^2}{1\cdot 2} + \&c. = h+\delta$$

or substituting for $f''(a)\frac{}{1\cdot 2}$ we have.

$$f'(a)(z+h) + h - \frac{f'(a)z}{z^2}\cdot(z+h)^2 = h+\delta$$

$$f'(a)\left\{z+h - \frac{(z+h)^2}{z}\right\} = h+\delta - h\cdot\left(\frac{z+h}{z}\right)^2$$

$$-zf'(a) = \frac{z^2(h+\delta) - h(z+h)^2}{(z+h)h}$$

similarly. changing $h$ into $h'$, $\delta$ into $\delta'$

$$-2f'(a) = \frac{z^2(h+\delta') - h(z+h')^2}{(z+h')h'}.$$

Equate the two values of $-2f'(a)$.

$$h'z^2(h+\delta)(z+h') - (z+h')h'h\{z^2+2hz+h^2\} =$$

$$hz^2(h+\delta')(z+h) - (z+h)hh\{z^2+2h'z+h'^2\}$$

which by reduction assumes the form.

$$z^2(h'\delta - h\delta') + (h'^2\delta - h^2\delta')z + hh'(h'-h) = 0.$$

$dy/dx = \dfrac{ay - 2bx}{3y - ax}$, find the value of $p$ when $x = y = 0$.

$$p = \frac{ay - 2bx}{3y - ax} = \frac{0}{0}.$$

$$= \frac{ap - 2b}{3p - a}$$

$$3p^2 - 2ap = -2b$$

$$p^2 - \frac{2}{3}ap + \frac{a^2}{9} = \frac{1}{9}\{a^2 - 6b\}.$$

$$p - \frac{a}{3} = \frac{1}{3}\sqrt{a^2 - 6b}.$$

$$\therefore p = \frac{1}{3}\{a \pm \sqrt{a^2 - 6b}\}.$$

Trace the curve whose $=^n$ is

$$x^4 + 2ax^2y - ay^3 = 0$$

we have $\left(\frac{dy}{dx}\right) = \frac{4x^3 + 4axy}{3ay^2 - 2ax^2} = \frac{0}{0}$ at the origin

$$= \frac{12x^2 + 4axp + 4ayp}{6ayp - 4ax} = \frac{0}{0}$$

$$= \frac{24x + 8ap}{6ap^2 - 4a} = \frac{4ap}{3ap^2 - 2a}.$$

$\therefore 3ap^3 - 6ap = 0$ or $p = 0$.

$p^2 = 2$ or $p = \pm\sqrt{2}$.

$x =$ gives $y = 0$. $\therefore$ the origin is a point on the curve. also $x =$ gives $dy =$ on the tangent at an infinite distance $\perp$ to the axis of $y$, also for

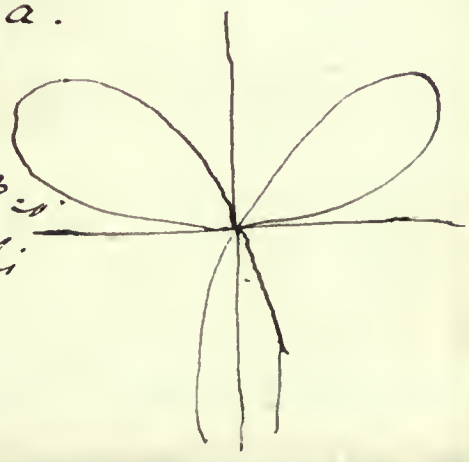
$\left(\frac{dy}{dx}\right) =$ $3ay^2 - 2ax^2 = 0$ or $\frac{y^2}{x^2} = \frac{2}{3}$

$\therefore \frac{y}{x} = \sqrt{\frac{2}{3}}$.

which value of $y$ substituted in the $=^n$ gives

$$x^4 + 2ax^2\sqrt{\tfrac{2}{3}} \cdot -\tfrac{2}{3}x^3 \cdot a\sqrt{\tfrac{2}{3}} = 0.$$

$$x + \tfrac{4}{3}a\sqrt{\tfrac{2}{3}} = 0. \quad \text{or } x = \pm\sqrt{\tfrac{2}{3}} \cdot \tfrac{4}{3}a.$$

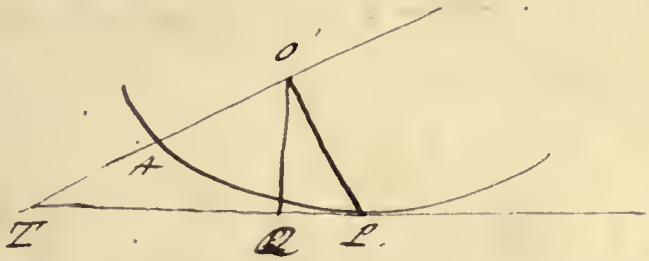


The curve is represented by the fig.

To find the max. value of $y$. we have $u = x^4 + 2axy^2 - ay^3 = 0$.
$du = 4x^3 + 4axy =$ &c. $x = 0$ or $y = -\frac{x^2}{a}$. substitute this
value of $y$ in the $=^n$ and $x = \pm a\sqrt{2}$, $\therefore y = 2a$.

Find the curve whose pole shall trace out a catenary when it rolls along a st line

O the pole. A L the curve.
O L = r. O R = p the ⊥ upon the tangent.




O L = O R sec ∠ O L = O R $\frac{ds}{dx}$. for L T is a tangent at L.
= p · $\frac{ds}{dx}$.

∴ if p = f(φ), being = ⁿ to A L, y = f(y · $\frac{ds}{dx}$) will be = ⁿ to the locus of the pole.

The = ⁿ to the catenary is $y = \frac{a}{2}\left\{ e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right\}$.

$$\frac{dy}{dx} = \frac{1}{2}\left\{ e^{\frac{x}{a}} - e^{-\frac{x}{a}} \right\}.$$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{1}{2}\cdot\left( e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right) = \frac{y}{a}.$$

$$\therefore y\frac{ds}{dx} = \frac{y^2}{a}$$

$$\text{but } y = \int \frac{y}{a} ds = \sqrt{\int y\, ds}.$$

$$\therefore p = \sqrt{ar}.$$

$p^2 = ar$ is the = ⁿ to A L which is the common ⊿.

The locus of the center of an ellipse rolling along a st. line is $dx = \frac{y^2}{\sqrt{a^2-y^2}\sqrt{y^2-b^2}}$.

The ⊥ from the centre onto the tangent is

$$p^2 = \frac{a^2b^2}{a^2+b^2-x^2}.$$

∴ the = n pedal locus is

$$y^2 = \frac{a^2b^2}{a^2+b^2-y^2\left(\frac{dx}{dy}\right)^2}.$$

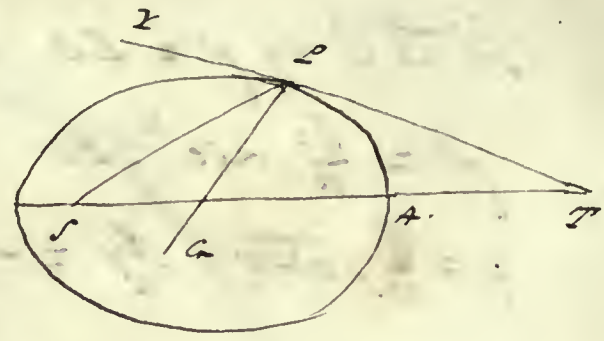
$$a^2+b^2-y^2\left(\frac{dx}{dy}\right)^2 = \frac{a^2b^2}{y^2}.$$

$$\frac{(a^2+b^2)y^2-a^2b^2}{y^2} = y^2\left(1+\left(\frac{dy}{dx}\right)^2\right).$$

$$-\frac{y^4+(a^2+b^2)y^2-a^2b^2}{y^4} = \left(\frac{dy}{dx}\right)^2.$$

$$\therefore dx = \frac{y^2}{\sqrt{a^2-y^2}\sqrt{y^2-b^2}}.$$

Le same otherwise.




$\cos SPZ = \sin SPG = e \cos STZ.$

$\sec SPZ = \frac{1}{e} \sec STZ = \frac{1}{e} \cdot \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$\therefore \sec^3 SPZ = \frac{1}{e^3} \cdot \left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}$

$= \rho \cdot \frac{d^2y}{e^3}.$

$\therefore \rho = \left(1-\frac{x^2}{a^2}\right)$

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$1 - \sin^2 SPG = \cos^2 SPG = \left\{1 - \frac{e^2}{1+\left(\frac{dy}{dx}\right)^2}\right\}$

$\sec^2 SPG = \frac{1+\left(\frac{dy}{dx}\right)^2}{1-e^2+\left(\frac{dy}{dx}\right)^2} = \left(1+\left(\frac{dy}{dx}\right)^2\right) \frac{1}{\frac{b^4}{a^2y^2}}.$

$\sec^3 SPG = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}.$

$\frac{a^3}{b^6} \sec^3 SPG = \rho \cdot \frac{d^2y}{dx^2}$

$= \frac{b^4}{a^2y^3}$

$\therefore \frac{b^6}{a^3y^3} \sec^3 SPG = \frac{b^4}{a^2y^3} \rho.$

$\frac{b^2}{a} \sec^3 SPG = \rho.$

In the ⊙ if $\rho = CO$. Then

$$\rho = \frac{b^2}{a} \cdot \sec^3 \angle CO.$$




$$y = \frac{b}{a}\sqrt{a^2 - x^2}; \quad \frac{dy}{dx} = -\frac{b^2 x}{a^2 y}. \quad \frac{d^2y}{dx^2} = -\frac{b^4}{a^2 y^3}.$$

$$\therefore \rho = -\frac{1}{\frac{d^2y}{dx^2}}\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}} = \frac{a^2y^3}{b^4}\left\{\frac{a^4y^2 + b^4x^2}{a^2y}\right\}^{\frac{3}{2}}$$

$$= \frac{1}{ab^4}\cdot\left(b^4 + (a^2 - b^2)y^2\right)^{\frac{3}{2}}.$$

Also $\tan \angle CO = \frac{dy}{dx} = \frac{e\sin\theta}{1 - e\cos\theta}$.

$$\therefore \sec^2 \angle CO = \frac{1 - 2e\cos\theta + e^2}{1 - 2e\cos\theta + e^2\cos^2\theta} = \frac{1 - 2\frac{\sqrt{a^2-b^2}}{a}\cdot\frac{ae+x}{a+ex} + \frac{a^2-b^2}{a^2}}{1 - 2\frac{\sqrt{a^2-b^2}}{a}\frac{ae+x}{a+ex} + \frac{a^2-b^2}{a^2}\left(\frac{ae+x}{a+ex}\right)^2}$$

$$= \frac{a^4 - (a^2-b^2)x^2 - 2(a^2-b^2)a^2 - 2\cdot(a^2-b^2)^{\frac{3}{2}}x + (a^2-b^2)\left\{a^2 + 2\sqrt{a^2-b^2}\cdot x + \frac{a^2}{a}\right.}{a^4 - (a^2-b^2)x^2 - 2(a^2-b^2)a^2 - 2\cdot(a^2-b^2)^{\frac{3}{2}}x + (a^2-b^2)\left\{a^2-b^2+2\sqrt{a^2-b^2}\cdot x\right.}$$

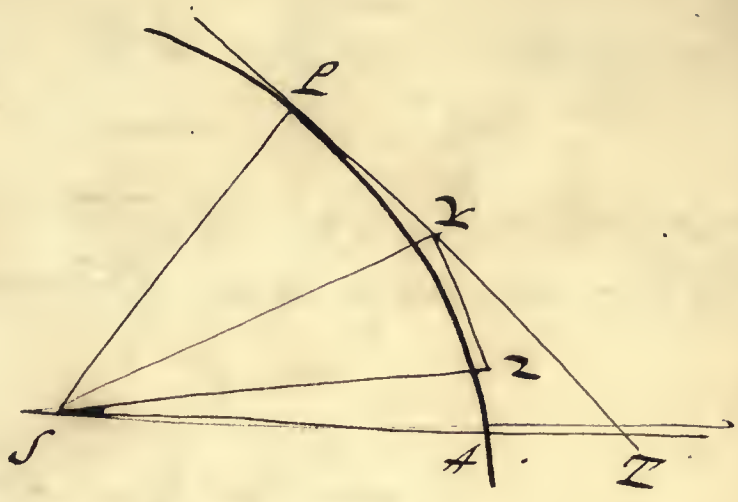
$$= \frac{(a^2-b^2)b^2x^2 - a^4b^2}{(a^2-b^2)(a^4+b^2a^2) - a^6} = \frac{(a^2-b^2)(b^2y^2) - a^2b^2}{(a^2-b^2)(a^2b^2) - a^4}$$

$$= \frac{b^4 + (a^2-b^2)y^2}{b^4}.$$

Also $\rho = \frac{1}{ab^4}\left\{b^4 + (a^2-b^2)y^2\right\}^{\frac{3}{2}} = \frac{b^2}{a}\cdot\left\{\frac{\sqrt{b^4 + (a^2-b^2)y^2}}{b^2}\right\}^{\frac{3}{2}}$.

$$= \frac{b^2}{a}\sec^3 \angle CO.$$

If $f(\rho, \rho') = 0$ be the $n^{th}$ Pedal
Spiral, then $f\left(\rho', \frac{\rho''^2}{p'}\right) = 0$
is the $n^{th}$ Pedal locus of $Z$.




Because $SX^2 = SP \cdot SZ$.

or $\rho p' = p^2 = \rho''^2$

$$\left.\begin{aligned} \therefore \rho &= \frac{\rho''^2}{p'} \\ p &= \rho' \end{aligned}\right\}$$

or $f\left(\rho', \frac{\rho''^2}{p}\right) = 0$ is the $n^{th}$ Pedal locus of $Z$.



Trace the curve whose $=^n$ is $\frac{y}{a} = \sqrt{\frac{a}{x} - 1}$, shewing there is a point of contrary flexure if $x = \frac{3a}{4}$.

since $x = 0$ makes $y = \infty$. the axis of $y$ is an asymptote also if $x$ be a negative quantity, or a +ve quantity $> a$. $y$ is impossible.

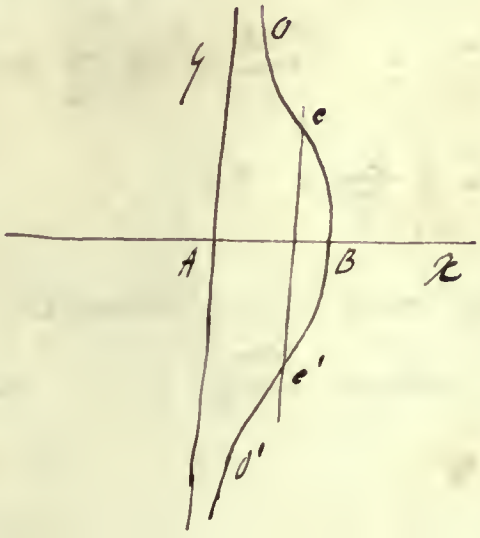
$$\frac{y}{a} = \sqrt{\frac{a-x}{x}} \text{ gives } \frac{dy}{a} = -\frac{1}{2} \frac{a}{x^{\frac{3}{2}}\sqrt{a-x}}.$$

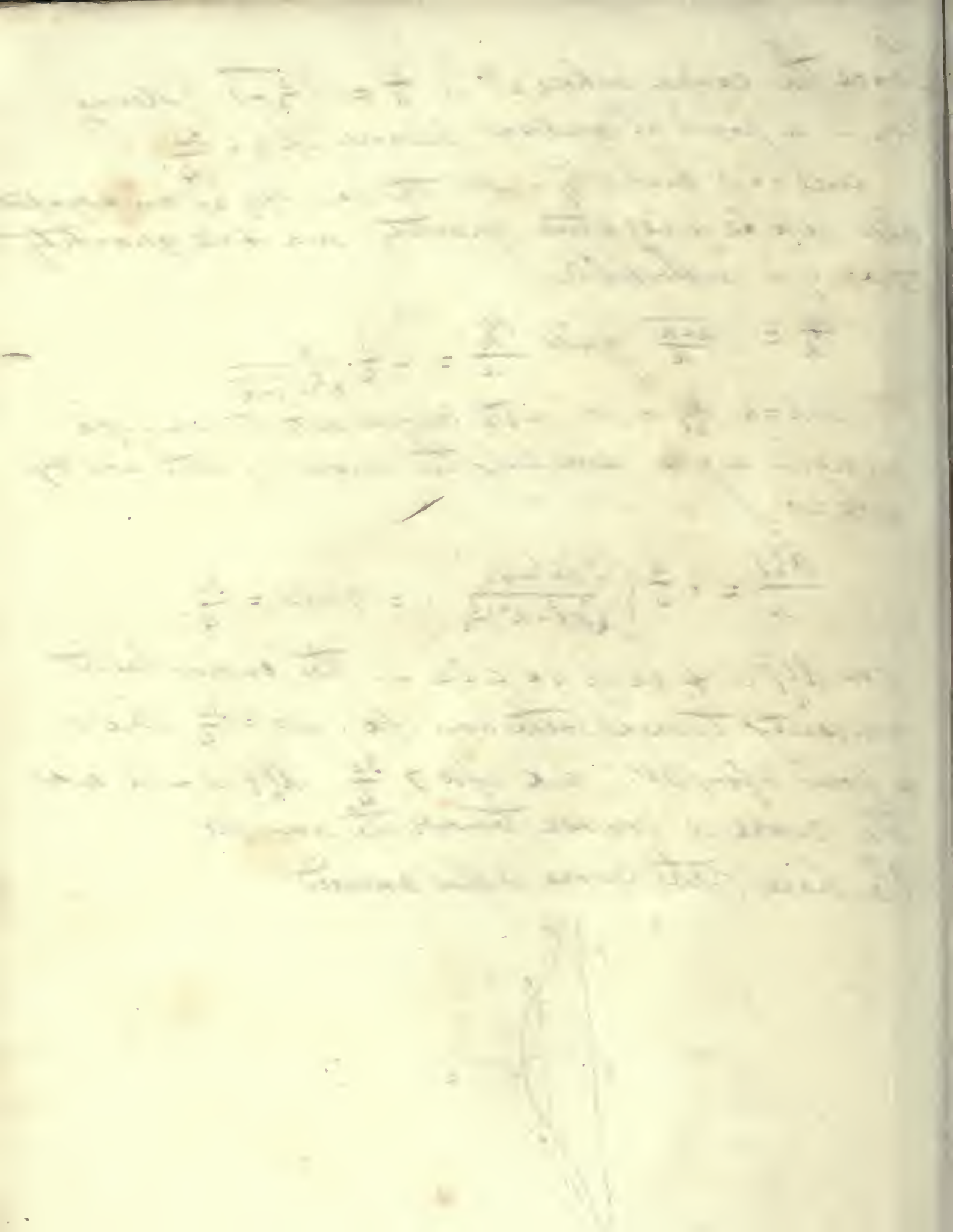
$\therefore$ if $x = a$. $\frac{dy}{dx} = \infty$, or the curve cuts the axis of $x$ at right angles. similarly the curve is $\parallel$ to the axis of $y$ if $x = 0$.

$$\frac{d^2y}{a} = +\frac{a}{4}\left\{\frac{3ax^2 - 4x^3}{(ax^3 - x^4)^{\frac{3}{2}}}\right\} = 0 \text{ if } x = \frac{3a}{4}.$$

Now $\frac{d^2y}{dx^2}$ is + ve if $4x < 3a$ $\therefore$ the curve has its convexity turned to the axis of $x$. if $x = \frac{3a}{4}$. there is a point of inflexion, and if $x > \frac{3a}{4}$. $\frac{d^2y}{dx^2}$ is $-$ ve, and the curve is concave towards the axis of $x$.

The shape of the curve is that annexed





If $y^4 - x^5 + x^4 + 3x^2y^2 = 0$ show that the origin is a conjugate point.

---

$$y^4 - x^5 + x^4 + 3x^2y^2 = 0$$

$$(4y^3 + 6x^2y)\frac{dy}{dx} = 5x^4 - 4x^3 - 6xy^2$$

$$\frac{dy}{dx} = \frac{5x^4 - 4x^3 - 6xy^2}{4y^3 + 6x^2y} = \frac{0}{0} \quad \begin{matrix} 4x=0 \\ y=0 \end{matrix}$$

$$p = \frac{20x^3 - 12x^2 - 12yp}{12y^2p + 12xy + 6x^2p} = \frac{0}{0}.$$

$$= \frac{60x^2 - 24x - 12yp - 12xp^2}{24yp^2 + 12xp + 12y + 12xp} = \frac{0}{0}$$

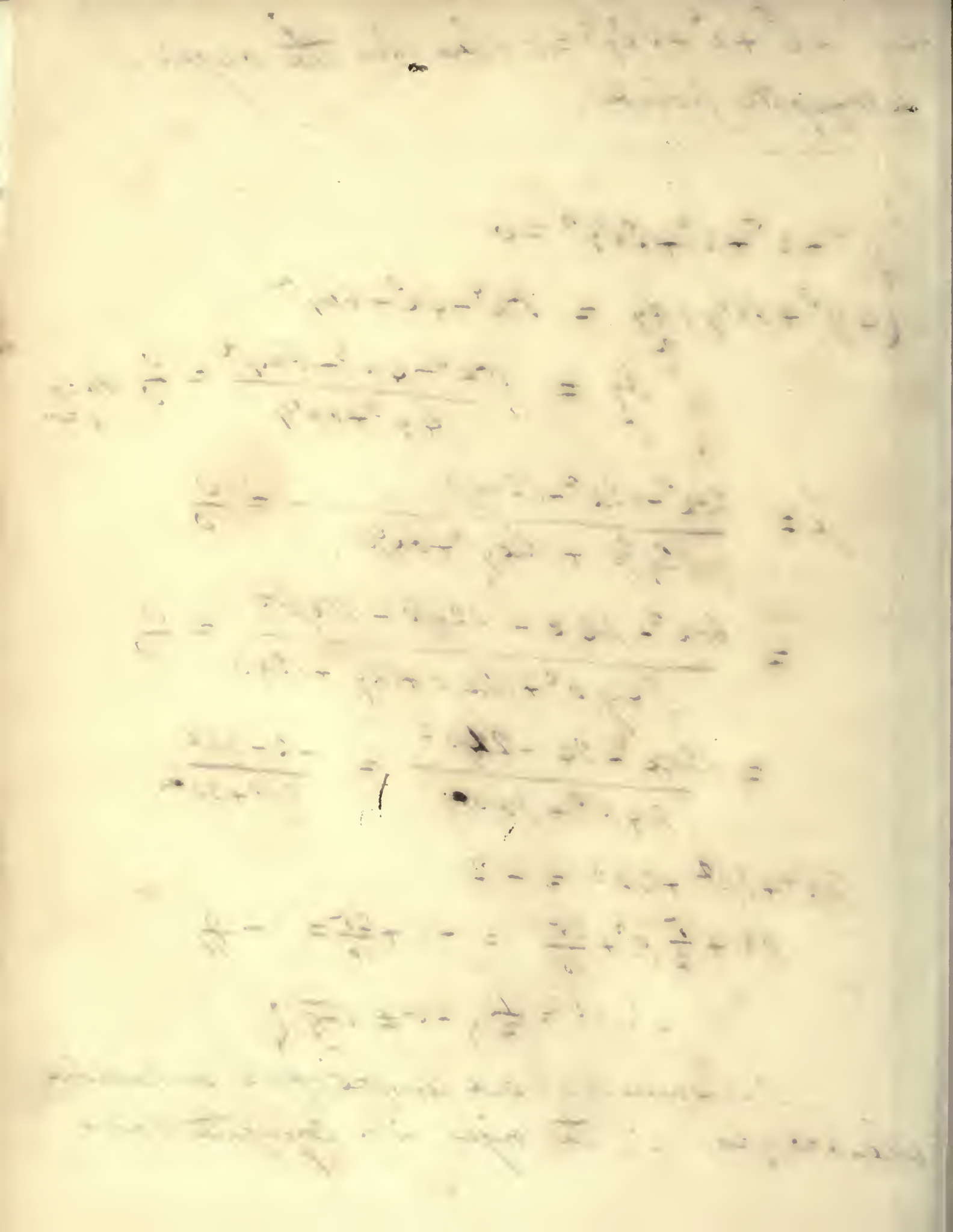
$$= \frac{120x - 24 - 24p^2}{24p^3 + 36p} = \frac{-2 - 2p^2}{2p^3 + 3p}$$

$$2p^4 + 3p^2 + 2p^2 = -2$$

$$p^4 + \frac{5}{2}p^2 + \frac{25}{16} = -1 + \frac{25}{16} = -\frac{9}{16}.$$

$$\therefore p^2 = \frac{1}{4}\left\{-5 \pm \sqrt{-9}\right\}.$$

$\therefore$ the value of $p^2$ and therefore of $p$ is impossible when $x=0, y=0$ $\therefore$ the origin is a conjugate point



Baly

# Integral Calculus.

To find the area of the evolute of a ⊙

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{\beta}\right)^{\frac{2}{3}} = 1$$

$$\frac{y}{\beta} = \left(1 - \left(\frac{x}{a}\right)^{\frac{2}{3}}\right)^{\frac{3}{2}} \quad \therefore \quad y = \frac{\beta}{a}\cdot\left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{3}{2}}.$$

$$\text{Let } x^{\frac{2}{3}} = z^2 \quad \therefore \quad \tfrac{2}{3}\log x = 2\log z$$

$$dx = 3\frac{x}{2}\,dz = 3z^2\,dz.$$

$$\therefore y\,dx = \frac{3\beta}{a}\left\{a^{\frac{2}{3}} - z^2\right\}^{\frac{3}{2}} z^2\,dz.$$

$$A = \frac{3\beta}{a}\int_z \left(a^{\frac{2}{3}} - z^2\right)^{\frac{3}{2}} z^2\,dz = \frac{3\beta}{a}\left\{-\int_z \left(a^{\frac{2}{3}} - z^2\right)^{\frac{5}{2}} + a^{\frac{2}{3}}\int_z \left(a^{\frac{2}{3}} - z^2\right)^{\frac{3}{2}}\right.$$

$$\text{Now } \int_z \left(a^{\frac{2}{3}} - z^2\right)^{\frac{5}{2}} = \frac{2}{5}\cdot\left(a^{\frac{2}{3}} - z^2\right)^{\frac{5}{2}} + \frac{5}{5}\int_z z^2\left(a^{\frac{2}{3}} - z^2\right)^{\frac{3}{2}}$$

$$= \frac{2\cdot\left(a^{\frac{2}{3}} - z^2\right)^{\frac{5}{2}}}{6} + \frac{5a^{\frac{2}{3}}}{6}\cdot\int_z \left(a^{\frac{2}{3}} - z^2\right)^{\frac{3}{2}}.$$

$$\therefore -\int_z \left(a^{\frac{2}{3}} - z^2\right)^{\frac{5}{2}} + a^{\frac{2}{3}}\int_z \left(a^{\frac{2}{3}} - z^2\right) = \frac{a^{\frac{2}{3}}}{6}\cdot\int_z \left(a^{\frac{2}{3}} - z^2\right)^{\frac{3}{2}}.$$

$$= \frac{a^{\frac{2}{3}}}{6}\left\{\frac{2\left(a^{\frac{2}{3}} - z^2\right)^{\frac{3}{2}}}{4} + \frac{3a^{\frac{2}{3}}}{4}\cdot\int_z \left(a^{\frac{2}{3}} - z^2\right)^{\frac{1}{2}}\right\}.$$

$$= \frac{a^{\frac{4}{3}}}{8}\cdot\int_z \left(a^{\frac{2}{3}} - z^2\right)^{\frac{1}{2}} \text{ from } z = a^{\frac{1}{3}} \text{ to } z = -a^{\frac{1}{3}}.$$

$$= \frac{a^{\frac{4}{3}}}{8}\left\{\frac{z}{2}\left(a^{\frac{2}{3}} - z^2\right)^{\frac{1}{2}} + \frac{a^{\frac{2}{3}}}{2}\sin^{-1}\frac{z}{a^{\frac{1}{3}}}\cdot\right\}.$$

$$= \frac{a^2}{8}\cdot\frac{\pi}{2} \text{ from } z = a^{\frac{1}{3}} \text{ to } z = -a^{\frac{1}{3}}.$$

Take twice of this as the curve is symmetrical with respect to the axis and multiply by $\frac{3\beta}{a}$. and we have the whole area of the curve

$$= 2\cdot\frac{3\beta}{\alpha}\cdot\frac{\alpha^2\pi}{8\cdot 2} = \frac{3\alpha\beta\pi}{8}.$$

$$f(m) = \int_0^\pi \frac{\cos m\theta}{1 - e\cos\theta}$$ where $m$ is an integer.

$$f(m+2) = \frac{2}{e} \cdot f(m+1) - f(m).$$

$$\cos^2\theta = \frac{1}{2}\left(x + \frac{1}{x}\right)$$

$$\frac{\cos m\theta}{1 - e\cos\theta} = \frac{\cos m\theta}{1 - \frac{e}{2}\left(x + \frac{1}{x}\right)} = \cos m\theta \left\{ 1 + \frac{e}{2}\left(x + \frac{1}{x}\right) + \frac{e^2}{4}\left(x^2 + \frac{1}{x^2} + 2\right) + \frac{e^3}{8}\left\{x^3 + \frac{1}{x^3} + 3\left\{x + \frac{1}{x}\right\}\right\} + \&c.\right.$$

$$= \cos m\theta \left\{ 1 + e\cos\theta + \frac{e^2}{2}\left\{\cos 2\theta + 1\right\} + \frac{e^3}{4}\left\{\cos 3\theta + 3\cos\theta\right\} + \&c.\right.$$

$$= \int_0^\pi \left\{ \cos m\theta + \frac{e}{2}\left\{\cos(m+1)\theta + \cos(m-1)\theta\right\} + \frac{e^2}{2}\left\{\cos m\theta + \frac{1}{2}\left\{\cos(m+2)\theta + \cos(m-2)\theta\right\}\right\} \right.$$
$$\left. + \frac{e^3}{8}\left\{\cos(m+3)\theta + \cos(m-3)\theta + 3\cos(m+1)\theta + 3\cos(m-1)\theta\right\} + \&c. \right\}$$

$$f(m+2) = \int_0^\pi \left\{ \cos(m+2)\theta + \frac{e}{2}\left\{\cos(m+3)\theta + \cos(m+1)\theta\right\} + \frac{e^2}{2}\left\{\cos(m+2)\theta + \frac{1}{2}\left\{\cos(m+4)\theta + \cos m\theta\right\}\right\} \right.$$
$$\left. + \&c. \right\}$$

$$f(m+1) = \int_0^\pi \left\{ \frac{2}{e}\cos m\theta + \cos(m+2)\theta + \cos(m+3)\theta + \&c \right\} \cos m\theta + \frac{1}{2}\left\{\cos(m+3)\theta + \cos(m-1)\theta\right\}.$$

$$\frac{2}{e} \cdot f(m+1) - f(m) =$$

$$\int_0^\pi \left\{ \frac{2}{e}\cos m\theta + \cos(m+2)\theta + \frac{e}{2}\left(\cos(m+1)\theta + \cos(m+3)\theta\right\} + \cancel{\cos(m-1)\theta} \right.$$
$$= \int_0^\pi \left\{ \cos(m+2)\theta + \frac{e}{2}\left\{\cos(m+1)\theta + \cos(m+3)\theta\right\} + \&c. \right.$$

for $\int_0^\pi \cos m\theta = 0$

$$= f(m+2)\theta.$$

$$\int_x \frac{\sin(n\cot^{-1}x)}{(1+x^2)^{\frac{n}{2}}} = -\frac{1}{n}\int_x \sin(n\cot^{-1}x)\cdot\frac{1}{(1+x^2)^{\frac{n-2}{2}}}$$

$$= -\frac{1}{n}\cdot\cos(n\cot^{-1}x)\frac{1}{(1+x^2)^{\frac{n-2}{2}}} - \frac{n-2}{n}\int_x \cos(n\cot^{-1}x)\frac{x}{(1+x^2)^{\frac{n}{2}}}$$

$$\int_x \frac{n-2}{n}\cos(n\cot^{-1}x)\frac{x}{(1+x^2)^{\frac{n}{2}}} = -\frac{n-2}{n^2}\int_x \frac{\cos(n\cot^{-1}x)}{(1+x^2)}\cdot\frac{x}{(1+x^2)^{\frac{n-2}{2}}}$$

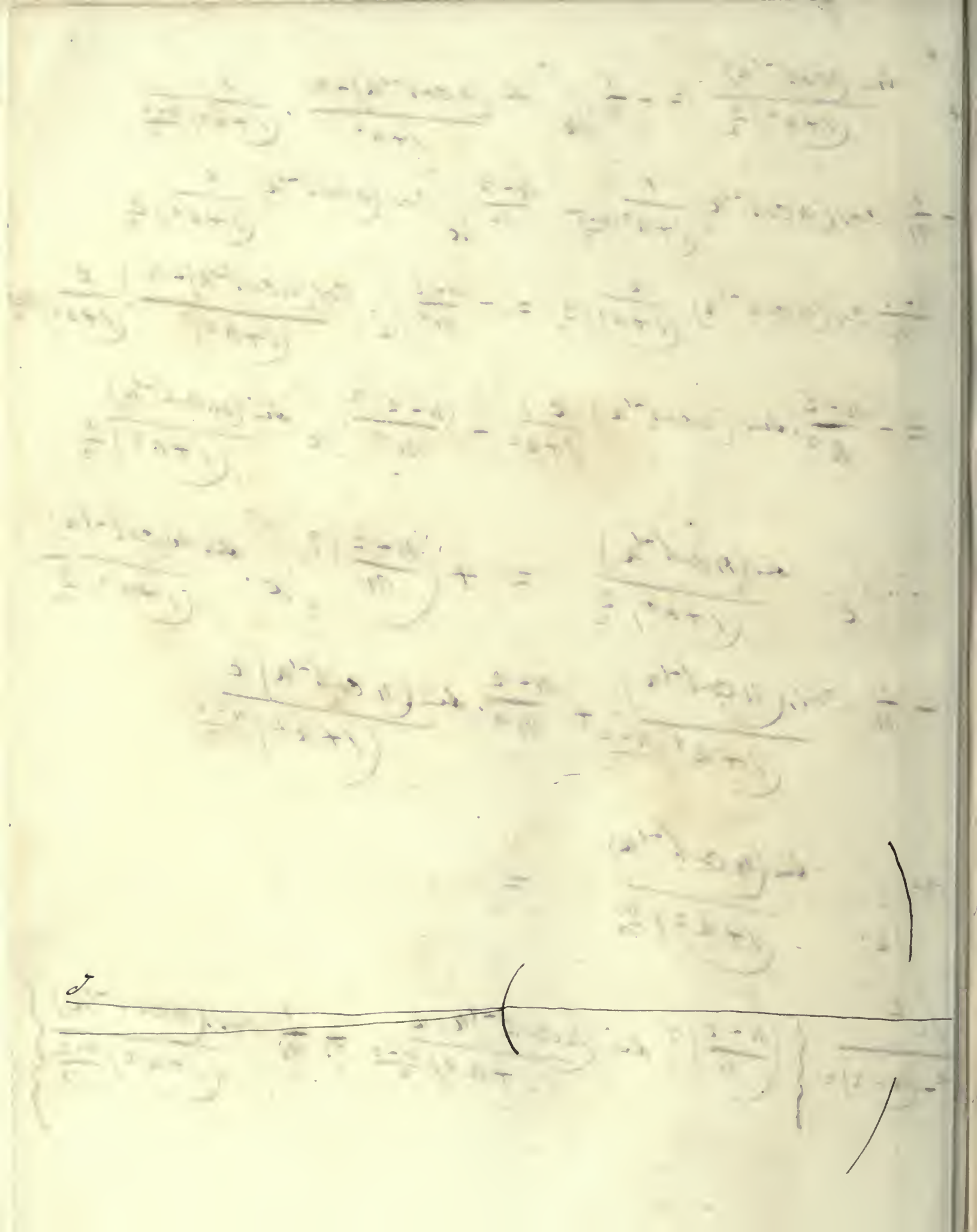
$$= -\frac{n-2}{n^2}\cdot\sin(n\cot^{-1}x)\frac{x}{1+x^2} - \frac{(n-2)^2}{n^2}\cdot\int_x \frac{\sin(n\cot^{-1}x)}{(1+x^2)^{\frac{n}{2}}}$$

$$\therefore \int_x \frac{\sin(n\cot^{-1}x)}{(1+x^2)^{\frac{n}{2}}} = +\left(\frac{n-2}{n}\right)^2\int_x \frac{\sin n(\cot^{-1}x)}{(1+x^2)^{\frac{n}{2}}}$$

$$-\frac{1}{n}\cos(n\cot^{-1}x)\frac{}{(1+x^2)^{\frac{n-2}{2}}} + \frac{n-2}{n^2}\cdot\frac{\sin(n\cot^{-1}x)x}{(1+x^2)^{\frac{n-2}{2}}}$$

$$\text{or} \int_x \frac{\sin(n\cot^{-1}x)}{(1+x^2)^{\frac{n}{2}}} =$$

$$\frac{x^2}{n^2-(n-2)^2}\left\{\left(\frac{n-2}{n}\right)^2\sin\frac{(n\cot^{-1}x)x}{(1+x^2)^{\frac{n-2}{2}}} - \frac{1}{n}\cos\frac{(n\cot^{-1}x)}{(1+x^2)^{\frac{n-2}{2}}}\right\}$$



$$\int \left(\log x\right)^n \cdot \left(x^n + \frac{1}{n}x^{n-1} + \frac{1}{n}\right) = \int x \cdot \frac{d}{dx}\left(\log x\right)^n$$

$$= x \cdot \left(\log x\right)^n - n \int \frac{x}{x} \cdot \left(\log x\right)^{n-1}$$

continuing therefore in this method shall come to

$$\int \phi(x) \log x = \psi(x) \cdot \log x - \int \frac{\psi(x)}{x}$$

$$= \psi x \log(x) - \chi(x).$$

$$\int x^2 \cdot \log \frac{a + \sqrt{a^2 - x^2}}{x} = \frac{x^3}{3} \cdot \log\left(\frac{a + \sqrt{a^2 - x^2}}{x}\right) - \frac{1}{3}\int x^3 \left\{ \frac{\frac{-x^2}{\sqrt{a^2 - x^2}} - \left(a + \sqrt{a^2 - x^2}\right)}{x\left(a + \sqrt{a^2 - x^2}\right)} \right\}$$

$$= \frac{x^3}{3} \log \frac{a + \sqrt{a^2 - x^2}}{x} + \frac{a}{3}\int \frac{x^2}{x \cdot \sqrt{a^2 - x^2}}$$

$$= \frac{x^3}{3} \log \frac{a + \sqrt{a^2 - x^2}}{x} + \frac{a}{3}\int \frac{x}{\sqrt{a^2 - x^2}} + \frac{a^3}{3}\int \frac{1}{x\sqrt{a^2 - x^2}}$$

$$= \frac{x^3}{3} \log \frac{a + \sqrt{a^2 - x^2}}{x} - \frac{a}{6} x\sqrt{a^2 - x^2} + \frac{a^3}{6} \sin^{-1} \frac{x}{a}.$$

$$\int \frac{1}{1 + \sin x} = \frac{1}{2}\int \frac{2 \cos x}{\cos x + \sin x} = \frac{1}{2}\int \left\{ \frac{-\sin x + \cos x + \cos x + \sin x}{\cos x + \sin x} \right\}$$

$$= \frac{1}{2} \log\left(\cos x + \sin x\right) + \frac{1}{2} x.$$

$$\int \frac{1 + \sin x}{1 + \cos x} = \frac{1}{2}\int \sec^2 \frac{x}{2} + \int \frac{\sin x}{1 + \cos x}.$$

$$= \tan \frac{x}{2} - \log_e\left(1 + \cos x\right).$$

$$\int \frac{x^m}{\sqrt{2ax-x^2}} = \int \frac{x^{m-\frac{1}{2}}}{\sqrt{2a-x}} = -2x^{m-\frac{1}{2}}\sqrt{2a-x} + (2m-1)\int x^{m-\frac{3}{2}}\sqrt{2a-x}.$$

$$\int x^{m-\frac{3}{2}}\sqrt{2a-x} = -\int \frac{x^{m-\frac{1}{2}}}{\sqrt{2a-x}} + 2a\int \frac{x^{m-\frac{3}{2}}}{\sqrt{2a-x}}$$

$$\int \frac{x^m}{\sqrt{2ax-x^2}} = -2x^{m-\frac{1}{2}}\sqrt{2a-x} - (2m-1)\int \frac{x^{m-\frac{1}{2}}}{\sqrt{2a-x}} + (2m-1)2a\int \frac{x^{m-\frac{3}{2}}}{\sqrt{2a-x}}$$

$$\text{or} \int \frac{x^{m-\frac{1}{2}}}{\sqrt{2a-x}} = -\frac{x^{m-\frac{1}{2}}\sqrt{2a-x}}{m} + \frac{(2m-1)a}{m}\int \frac{x^{m-\frac{3}{2}}}{\sqrt{2a-x}}.$$

$$\int \frac{x^m}{\sqrt{2ax-x^2}} = -\frac{x^{m-1}\cdot\sqrt{2ax-x^2}}{m} + \frac{(2m-1)a}{m}\int \frac{x^{m-1}}{\sqrt{2ax-x^2}}.$$

$$\int_x e^{ax} \cdot (\cos x)^n = \frac{1}{a} \cdot e^{ax} \cdot \cos x^n + \frac{n}{a} \cdot \int_x e^{ax} \cdot \cos x^{n-1} \sin x \cdot$$

$$\int_x e^{ax} \cos x^{n-1} \sin x = \frac{1}{a} e^{ax} \cdot \cos x^{n-1} \sin x - \frac{1}{a} \int_x e^{ax} \left\{ (n-1) \cos x^{n-2} \cdot x - \sin x^2 + \cos x^n \right\}$$

$$= \frac{1}{a} e^{ax} \cos x^{n-1} \sin x - \frac{1}{a} \int_x e^{ax} \left\{ -(n-1) \cos x^{n-2} (1 - \sin^2 x) - (n-1) \cos x^n \right\}$$

$$= \frac{1}{a} e^{ax} \cos x^{n-1} \sin x - \frac{n}{a} \int_x e^{ax} \cos x^n + \frac{n-1}{a} \int_x e^{ax} \cos x^{n-2}$$

$$\frac{a^2+n^2}{a^2} \int_x e^{ax} \cos x^n = \frac{e^{ax}}{a^2} \left\{ a \cos x^n + n \cos x^{n-1} \sin x \right\} + \frac{n(n-1)}{a^2} \int_x e^{ax} \cos x^{n-2}$$

$$\int_x e^{ax} \cos x^n = \frac{1}{a^2+n^2} e^{ax} \cdot \left( a \cos x^n + n \cos x^{n-1} \sin x \right) + \frac{n(n-1)}{a^2+n^2} \int_x e^{ax} \cos x^{n-2} \cdot$$

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$$\int_x \sqrt{a+b \tan^2 x} = \int_x \frac{a + b \sec^2 x - b}{\sqrt{a+b \tan^2 x}}$$

$$= \sqrt{b} \cdot \int \tan x \frac{1}{\sqrt{\frac{a}{b} + \tan^2 x}} + (a-b) \cdot \int_x \frac{\cos x}{\sqrt{a(1-\sin^2 x) + b \sin^2 x}}$$

$$= \sqrt{b} \cdot \log \left\{ \tan x + \sqrt{\frac{a}{b} + \tan^2 x} \right\} + \sqrt{b-a} \cdot \int \sin x \frac{1}{\sqrt{\frac{a}{b-a} + \sin^2 x}}$$

$$= \sqrt{b} \cdot \log \left\{ \tan x + \sqrt{\frac{a}{b} \cdot \tan^2 x} \right\} + \sqrt{b-a} \log \left\{ \sqrt{\frac{a}{b-a}} + \sqrt{\frac{a}{b-a} + \sin^2 x} \right\}$$

$$\frac{1}{(1+x^2)(a^2+b^2x^2)} = \frac{A}{1+x^2} + \frac{B}{a^2+b^2x^2}.$$

$$1 = A(a^2+b^2x^2) + B.(1+x^2).$$

$$\therefore Aa^2+B = 1. \quad Ab^2+B = 0.$$

$$\therefore A = \frac{1}{a^2-b^2} \; ; \; B = \frac{-b^2}{a^2-b^2}.$$

$$\int \frac{1}{(1+x^2)(a^2+b^2x^2)} = \frac{1}{a^2-b^2}\left\{\int \frac{1}{1+x^2} - b^2\int \frac{1}{a^2+b^2x^2}\right)$$

$$= \frac{1}{a^2-b^2}\left\{\tan^{-1}x - \frac{b}{a}.\tan^{-1}\frac{ab}{a}\right) + C.$$

$\because \tan 0 = 0$ given $C = 0$.

$$\therefore \int_0^x \frac{1}{(1+x^2)(a^2+b^2x^2)} = \frac{1}{a^2-b^2}\left\{\tan^{-1}x - \frac{b}{a}\tan^{-1}\frac{xb}{a}\right\}.$$

If $x = \infty$ we have

$$\int_0^\infty \frac{1}{(1+x^2)(a^2+b^2x^2)} = \frac{1}{a^2-b^2}\left\{\frac{\pi}{2} - \frac{b}{a}\frac{\pi}{2}\right\}.$$

$$= \frac{a-b}{a(a^2-b^2)}\frac{\pi}{2} = \frac{\pi}{2}.\frac{1}{a(a+b)}.$$

$$\int x \cdot \sin^{-1}\frac{\sqrt{a^2-x^2}}{\sqrt{b^2-x^2}} = x\sin^{-1}\frac{\sqrt{a^2-x^2}}{\sqrt{b^2-x^2}} + \sqrt{b^2-a^2}\int \frac{x^2}{x\sqrt{a^2-x^2}\cdot(b^2-x^2)}$$

$$\int_x^a \sin^{-1}\frac{\sqrt{a^2-x^2}}{\sqrt{b^2-x^2}} = \sqrt{b^2-a^2}\int_x^a \frac{x^2}{x\cdot\sqrt{a^2-x^2}(b^2-x^2)} \qquad +C.$$

$$\int x \frac{x^2}{\sqrt{a^2-x^2}(b^2-x^2)} = -\int x \frac{b^2-x^2-b^2}{\sqrt{a^2-x^2}\cdot(b^2-x^2)} = -\int x \frac{1}{\sqrt{a^2-x^2}} + b^2\int x \frac{1}{\sqrt{a^2-x^2}(b^2-x^2)}$$

$$= -\sin^{-1}\frac{x}{a} + \frac{b}{\sqrt{b^2-a^2}}\cdot\tan^{-1}\frac{b\sqrt{a^2-x^2}}{x\sqrt{b^2-a^2}} + C.$$

Let $x=0$ and the integral becomes.

$$0 + \frac{b}{\sqrt{b^2-a^2}}\cdot\frac{\pi}{2} + C. \quad \therefore \quad C = -\frac{b}{\sqrt{b^2-a^2}}\cdot\frac{\pi}{2}.$$

Let $x=a$ and the integral becomes.

$$-\frac{\pi}{2} + C = 0 \quad \therefore \quad C = \frac{\pi}{2} =$$

$$\therefore \int_x^{\sqrt{b^2-a^2}} \frac{x^2}{\sqrt{a^2-x^2}\cdot(b^2-x^2)} = \frac{\pi}{2}\left(b - \sqrt{b^2-a^2}\right)$$

$$\int_x \frac{\alpha x^2+\beta}{x^4+2fx^2+g^2}\cdot = \alpha\int_x \frac{x^2}{x^4+2fx^2+g^2} + \beta\int_x \frac{1}{x^4+2fx^2+g^2}$$

$$\int_x \frac{1}{x^4+2fx^2+g^2} = \frac{1}{g^2}\int_x \frac{1}{\frac{x^4}{g^2}+\frac{2f}{g^2}x^2+1} = \frac{1}{g^{\frac{3}{2}}}\int_z \frac{1}{z^4+\frac{2f}{g}z^2+1}.$$

If $f>g$, $z^4+\frac{2f}{g}z^2+1$ may be reduced to two real factors

$(z^2+a^2)(z^2+b^2)$.

$$\therefore \int_z \frac{1}{z^4+\frac{2f}{g}z^2+1} = \int_z \frac{A}{z^2+a^2} + \int_z \frac{B}{z^2+b^2}.$$

If $f<g$. $z^4+\frac{2f}{g}z^2+1 = \left(z^2-2z\cos\frac{\theta}{2}+1\right)\left(z^2-2z\cos\frac{3\theta}{2}+1\right)$.

Let $\left(z^2-2z\cos\frac{\theta}{2}+1\right) = (z-a)(z-b)$

$\therefore$ The partial fractions are $\frac{A}{z-a}+\frac{B}{z-b}$.

$$A = \frac{1}{4(a^2-\cos\frac{\theta}{2})a} = \frac{1}{4\sqrt{-1}\cdot\sin\theta\cdot a} = \frac{b}{4\sqrt{-1}\cdot\sin\theta}.$$

$$B = \frac{-a}{4\sqrt{-1}\sin\theta}.$$

$\therefore$ The partial fractions

$$\frac{1}{2\sin\theta}\cdot\frac{x\sin\frac{\theta}{2}}{x^2-2x\cos\frac{\theta}{2}+1} = \frac{\sin\frac{\theta}{2}}{2\sin\theta}\cdot\frac{x-\cos\frac{\theta}{2}+\cos\frac{\theta}{2}}{x^2-2x\cos\frac{\theta}{2}+1}$$

$\therefore$ The whole integral is

$$\frac{\sin\frac{\theta}{2}}{\sin\theta}\cdot\log\left(x^2-2x\cos\frac{\theta}{2}+1\right) - \frac{1}{4}\tan^{-1}\left(\frac{x-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right)$$

$$+\frac{\sin\frac{3\theta}{2}}{\sin\theta}\log\left(x^2-2x\cos\frac{3\theta}{2}+1\right) - \frac{1}{4}\tan^{-1}\frac{x-\cos\frac{3\theta}{2}}{\sin\frac{\theta}{2}}.$$

If $p = q$. above.

$$\int_x \frac{\alpha x^2+\beta}{x^4+2px^2+q^2} = \alpha \int_x \frac{x^2+q-q+\frac{\beta}{\alpha}}{(x^2+q)^2} = \alpha \int_x \frac{1}{x^2+q} + \beta - \alpha q \int_x \frac{1}{(x^2+q)^2}$$

$$\int x \cdot \frac{1}{(x^2+q)^2} = \frac{x}{2q(x^2+q)} + \frac{1}{2q} \int_x \frac{1}{x^2+q}.$$

$$\therefore \int_x \frac{\alpha x^2+\beta}{x^4+2px^2+q^2} = \beta - \alpha q \left\{ \frac{x}{2q(x^2+q)} \right\} + \frac{\beta+\alpha q}{2q^{\frac{3}{2}}} \cdot \tan^{-1} \frac{x}{\sqrt{q}}.$$

---

$$\int x \cdot \frac{\alpha+\beta x}{(c^2-x^2)\sqrt{a+bx}} = \frac{\alpha-c\beta}{2c} \int_x \frac{1}{(c+x)\sqrt{a+bx}} + \frac{\alpha+c\beta}{2c} \int_x \frac{1}{(c-x)\sqrt{a+bx}}.$$

$$= \frac{\alpha-c\beta}{c\sqrt{bc-a}} \cdot \tan^{-1} \sqrt{\frac{a+bx}{bc-a}}$$

$$+ \frac{\alpha+c\beta}{c\sqrt{bc+a}} \cdot \tan^{-1} \frac{\sqrt{a+bx}}{\sqrt{bc+a}}.$$

---

$$\int_x \sqrt{(a-x)(x-c)} = \int_x \sqrt{-ac+(a+c)x-x^2} = \int_x \sqrt{\left(\frac{a-c}{2}\right)^2 - \left\{\frac{a+c}{2}-x\right\}^2} = \int_x \sqrt{\left(\frac{a-c}{2}\right)^2 - \left\{x-\frac{a+c}{2}\right\}^2}$$

$$= \frac{1}{2}\left(x-\frac{a+c}{2}\right)\sqrt{(a-x)(x-c)} + \frac{1}{2}\left\{\frac{a-c}{2}\right\}^2 \cdot \tan^{-1}\left(\frac{2x-(a+c)}{a-c}\right)$$

$$\int \sin x \sqrt{1-c^2\sin^2 x}\,dx = -\cos x \sqrt{1-c^2\sin^2 x} + \int_x \frac{-c^2 \sin x \cos^2 x}{\sqrt{1-c^2\sin^2 x}}$$

$$= -\cos x \sqrt{1-c^2\sin^2 x} - \int_x \sin x \left\{ \frac{c^2-1+1-c^2\sin^2 x}{\sqrt{1-c^2\sin^2 x}} \right\}.$$

$$= -\frac{1}{2}\cos x \sqrt{1-c^2\sin^2 x} + \frac{1}{2}(1-c^2)\int_x \frac{\sin x}{\sqrt{1-c^2\sin^2 x}}.$$

$$\int_x \frac{\sin x}{\sqrt{1-c^2\sin^2 x}} = \frac{1}{c^2}\int_x \frac{\sin x}{\sqrt{\frac{1}{c^2}-1+(\cos x)^2}} = -\frac{1}{c^2}\cdot \log\left\{ \cos x + \sqrt{\frac{1-c^2}{c^2}+\cos^2 x}\right\} + C$$

$$\int_x \sin x \sqrt{1-c^2\sin^2 x} = -\frac{1}{2}\cos x \sqrt{1-c^2\sin^2 x} - \frac{1}{2}\,\frac{1-c^2}{c^2}\log\left\{ \cos x + \sqrt{\frac{1-c^2}{c^2}+\cos^2 x}\right\}$$

$$= -\frac{1}{2}\left\{ \cos x \sqrt{1-c^2\sin^2 x} + \frac{1-c^2}{c^2}\log\left\{ \cos x + \sqrt{\frac{1-c^2}{c^2}+\cos^2 x}\right\}\right\}$$

---

$$\int_t t^{2n} e^{-a^2t^2} = -\int_t -t e^{-a^2t^2}\cdot t^{2n-1} = -\frac{1}{2a^2}\cdot e^{-a^2t^2}\cdot t^{2n-1} + \frac{2n-1}{2a^2}\int_t t^{2n-2} e^{-a^2t^2} + C.$$

$$\int_t t^{2n} e^{-a^2t^2} = -\frac{1}{2a^2}\cdot e^{-a^2t^2}\cdot t^{2n-1} - \frac{2n-1}{2^2 a^4} e^{-a^2t^2} t^{2n-3} - \cdots$$

$$= -e^{-a^2t^2}\left\{ \frac{t^{2n-1}}{2a^2} + \frac{(2n-1)t^{2n-3}}{2^2a^4} + \frac{(2n-1)(2n-3)}{2^3a^6}\cdot t^{2n-5} - \cdots - \frac{(2n-1)\cdots 3\cdot 1}{2^n\cdot a^{2n}} t\right\} + \int \frac{(2n-1)\cdots 3\cdot 1}{2^n a^{2n}} \int t\, e^{-a^2t^2}.$$

$$\int_t^{-\infty}{}^{+\infty} t^{2n} e^{-a^2t^2} = \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{2^n a^{2n}} \int_t^{-\infty}{}^{+\infty} t\cdot e^{-a^2t^2}.$$

$$= \left\{ \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{2^n\cdot a^{2n+2}} \right\}\cdot \sqrt{\pi}.$$

$$\frac{1}{(x+a)^m (x+b)^n} = \frac{1}{(x+a)^{m+n} \left(1+\frac{b-a}{x+a}\right)^n}$$

$$= \frac{\left(\frac{1}{x+a}\right)^2 \left(\frac{b-a}{x+a}\right)^{-m-n+2}}{(b-a)^{m+n-2} \left(1+\frac{b-a}{x+a}\right)^n}$$

$$= \frac{\left(\frac{1}{x+a}\right)^2 \left(1+\frac{b-a}{x+a}-1\right)^{-m-n+2}}{(b-a)^{m+n-2} \left(1+\frac{b-a}{x+a}\right)^n}$$

$$= -\frac{1}{(b-a)^{m+n-1}} \frac{\frac{d}{dx}(u-1)^{-m-n+2}}{u^n}$$

$$= -\frac{1}{(b-a)^{m+n-1}} \cdot \frac{\frac{du}{dx}}{u^n} \left\{ u^{-m-n+2} - (m+n-2) u^{-m-n+1} + \dots \right.$$

$$= -\frac{1}{(b-a)^{m+n-1}} \left\{ \frac{\frac{du}{dx}}{u^{m+2n-2}} - (m+n-2) \frac{\frac{du}{dx}}{u^{m+2n-1}} + \dots \right\}$$

$$= -\frac{1}{(b-a)^{m+n-1}} \left\{ -\frac{1}{m+2n-1} \left( \frac{1}{u^{m+2n-1}} \right) + \frac{m+n-2}{m+2n} \cdot \frac{1}{u^{m+2n}} + \dots \right\}$$

$$\frac{1}{(x+a)^m}\cdot\frac{1}{(x+b)^n} = -\frac{1}{m-1}\cdot\frac{1}{(x+a)^{m-1}}\cdot\frac{1}{(x+b)^n} + \frac{n}{m-1}\cdot\frac{1}{(x+a)^{m-1}}\cdot\frac{1}{(x+b)^{n+1}}$$

$$\int\frac{1}{x+c} = \int\frac{1}{\sqrt{a-x}+\sqrt{b+x}}\cdot\frac{\sqrt{a-x}+\sqrt{b+x}}{\sqrt{a-x}-\sqrt{b+x}}\cdot a-x.$$

$$\sqrt{a-x}-\sqrt{b+x} = z.$$

$$\frac{-dx}{\sqrt{a+x}} =$$

$$\frac{\alpha x+\beta}{(x+p)(x+q)} = \frac{A}{x+p} + \frac{B}{x+q}$$

$$\alpha x+\beta = A(x+q) + B\cdot(x+p)$$

$$\alpha = (A+B) \quad \therefore \quad \alpha - A = B$$

$$\beta = Aq + Bp \;.\; = Aq + \alpha p - Ap.$$

$$= \alpha p + A(q-p)$$

$$\beta - \alpha p = A(q-p)$$

$$A = \frac{\beta-\alpha p}{q-p}.$$

$$B = \alpha - \frac{\beta-\alpha p}{q-p}.$$

$$= \frac{\alpha q - \alpha p - \beta + \alpha p}{q-p}$$

$$\therefore \int \frac{\alpha x+\beta}{(x+p)(x+q)\sqrt{a+bx+cx^2}}.$$

$$\frac{\beta-\alpha p}{q-p}\int_x \frac{1}{(x+p)\sqrt{a+bx+cx^2}}$$

$$+\frac{\alpha q-\beta}{q-p}\int_x \frac{1}{(x+q)\sqrt{a+bx+cx^2}}$$

$$\int_x \frac{1}{x^2+c} = \frac{1}{\sqrt{c}} \tan^{-1} x \cdot + C.$$

$x$ = upper limit $c = \infty \quad \therefore \int_0^\infty \frac{1}{x^2+c} = \frac{1}{\sqrt{c}} \frac{\pi}{2}.$

$$\int_x \frac{1}{x^4-4x+3} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{Cx+D}{x^2+2x+3}$$

$$\therefore \quad 1 = A.(x^2+2x+3) + B.(x-1)(x^2+2x+3) + (Cx+D).(x-1)^2.$$

$$x = 1.$$

$$1 = A.6 \quad \therefore \quad A = \tfrac{1}{6}.$$

$$-\frac{x^2}{6} - \frac{x}{3} - \frac{1}{2} = -\left\{\frac{x^2}{6} + \frac{x}{3} - \frac{1}{2}\right\} = -\left\{\frac{x}{6} + \frac{1}{2}\right\} = B(x^2+2x+3) + (Cx+D).x-1$$

$$\text{Let } x = 1$$

$$\therefore -\frac{2}{3} = 6B \quad \therefore \quad B = -\frac{1}{9}.$$

$$\frac{x}{6} - \frac{1}{2} + \frac{x^2}{9} + \frac{2x}{9} + \frac{1}{3} = \frac{1}{18}\left\{\frac{2x^2+x-3}{x-1}\right\} = \frac{2x+3}{18} = Cx+D.$$

$$\therefore \int_x \frac{1}{x^4-4x+3} = \frac{1}{6}\int_x \frac{1}{(x-1)^2} - \frac{1}{9}\int_x \frac{1}{x-1} + \frac{1}{18}\int_x \frac{2x+3}{x^2+2x+3}.$$

$$\int \frac{dx}{\sqrt[4]{a+bx^4}} = \frac{1}{36}\int \frac{1}{x^3}\cdot\frac{3bx^3}{\sqrt[4]{a+bx^4}} = \frac{1}{36}\cdot\frac{(a+bx^4)^{\frac{3}{4}}}{x^3} + \frac{3}{36}\int \frac{(a+bx^4)^{\frac{3}{4}}}{x^4}\cdot$$

$$a + 2bx^n = z^{-2n}.$$

$$\therefore -2n \cdot z^{-2n-1}\, dz = 2n \times bx^{n-1}\, dx.$$

$$dx = -\frac{z^{-2n-1}}{bx^{n-1}}.$$

$$\therefore x^{n-1} \cdot dx = -\frac{1}{b} \cdot \frac{1}{z^{2n+1}}$$

$$a + bx^n = \frac{z^{-2n} + a}{2}$$

$$\therefore \frac{x^{n-1} \cdot dx}{(a+bx^n)(a+2bx^n)^{\frac{1}{2n}}} = -\frac{1}{b} \cdot \frac{z^{-2n-1}\, dz}{z^{-1} \cdot \frac{z^{-2n}+a}{2}} =$$

$$= -\frac{2}{b} \cdot \frac{dz}{1 + az^{2n}}.$$

$$\therefore \int \frac{x^{n-1}}{x \cdot (a+bx^n)(a+2bx^n)^{\frac{1}{2n}}} = -\frac{2}{b} \int \frac{1}{z \cdot 1 + az^{2n}}.$$

$$\frac{b+cx}{x^3+a^3} = \frac{A}{x+a} + \frac{Bx+C}{x^2-ax+a^2}$$

$$b+cx = A.(x^2-ax+a^2) + (Bx+C)(x+a).$$

Let $x=-a$.

$$b-ac = A.\{3a^2\} \quad \therefore A = \frac{b-ac}{3a^2}.$$

$$b+x^2 = ax-a^2.$$

$$b+cx \quad \therefore B.ax - Ba^2 + Bax + Cx + Ca.$$

$$2B.a + C = c \quad \therefore C = c-2Ba.$$

$$b = Ca - Ba^2$$

$$= ac - 3Ba^2 \quad \therefore + \frac{b-ac}{3a^2} = -B.$$

$$C = \frac{3a^2c+2ba-2a^2c}{3a^2} = \frac{ac+2b}{3a}$$

$$\therefore \int \frac{b+cx}{x^3+a^3} = \frac{b-ac}{3a^2} \int \frac{1}{x+a} + \frac{1}{3a^2} \cdot \frac{(ac-b)x+(ac+2b)}{x^2-ax+b} \cdot .$$

$$\int \frac{1+c^2x^4}{1-c^2x^4}\cdot\frac{1}{\sqrt{1+px^2+c^2x^4}} = \frac{1}{2}\int_x \frac{1+cx^2}{1-cx^2}\cdot\frac{1}{\sqrt{1+px^2+c^2x^4}} + \frac{1}{2}\int_x \frac{1-cx^2}{1+cx^2}\cdot\frac{1}{\sqrt{1+px^2+c^2x^4}}$$

$$= \frac{1}{2}\left\{ \frac{1}{\sqrt{2c+p}}\cdot \log.\frac{1-cx^2}{x\sqrt{2c+p}+\sqrt{1+px^2+c^2x^4}} + \frac{1}{\sqrt{p-2c}}\cdot\log.\frac{1+cx^2}{x\sqrt{p-2c}+\sqrt{1+px^2+c^2x^4}}\right\}.$$

$$\frac{x^2}{1-c^2x^4}\cdot\frac{1}{\sqrt{1+px^2+c^2x^4}} = \frac{1}{4c}\int_x \frac{1+cx^2}{1-cx^2}\cdot\frac{1}{\sqrt{1+px^2+c^2x^4}} - \frac{1}{4c}\int_x \frac{1-cx^2}{1+cx^2}\frac{1}{\sqrt{1+px^2+c^2x^4}}\Bigg\}.$$

$$\int_x \frac{\sqrt{1+px^2+c^2x^4}}{1-c^2x^4} = \int_x \frac{1+px^2+c^2x^4}{1-c^2x^4}\cdot\frac{1}{\sqrt{1+px^2+c^2x^4}}$$

$$= \frac{2c+p}{4c}\cdot\int_x \frac{1+cx^2}{1-cx^2}\cdot\frac{1}{\sqrt{1+px^2+c^2x^4}} + \frac{2c-p}{4c}\cdot\int_x \frac{1+cx^2}{1+cx^2}\cdot\frac{1}{\sqrt{1+px^2+c^2x^4}}.$$

$$\frac{p+qx^2+rc^2x^4}{1-c^2x^4} = A.\frac{1+c^2x^2}{1-cx^2} + B.\frac{1-cx^2}{1+cx^2}.$$

$$\therefore p+qx^2+pc^2x^4 = A.(1+cx^2)^2 + B.(1-cx^2)^2.$$

$$\therefore p = A+B. \qquad \therefore p+\frac{q}{2c} = 2A. \quad A = \frac{2pc+q}{4c}$$

$$\frac{q}{2c} = A-B. \qquad \therefore$$

$$B = \frac{4pc-2pc-q}{4c} = \frac{2pc-q}{4c}$$

$$\int_x \frac{p+qx^2+pc^2x^4}{1-c^2x^4}\,dx = \frac{2pc+q}{4c}\int_x \frac{1+cx^2}{1-cx^2}\frac{1}{\sqrt{1+px^2+c^2x^4}} + \frac{2pc-q}{4c}\int_x \frac{1-cx^2}{1+cx^2}\frac{1}{\sqrt{1+px^2+c^2x^4}}.$$

$$\int_x \frac{1+cx^2}{1-cx^2}\frac{1}{\sqrt{1+px^2+c^2x^4}} = \int_x \frac{x^{-2}+c}{x^{-1}-cx}\cdot\frac{1}{\sqrt{x^{-2}+p+c^2x^2}} = -\int_x \frac{d_x.(x^{-1}-cx)}{x^{-1}-cx}\cdot\frac{1}{\sqrt{(x^{-1}-cx)^2+2c+p}}.$$

$$= \frac{1}{\sqrt{2c+p}}.\log.\frac{x^{-1}-cx}{\sqrt{2c+p}+\sqrt{x^{-2}+p+c^2x^2}}.$$

Similarly we have.

$$\frac{1-cx^2}{1+cx^2}\frac{1}{\sqrt{1+px^2+c^2x^4}} = \int_x \frac{x^{-2}-1}{x^{-1}+cx}\cdot\frac{1}{\sqrt{x^{-2}+p+c^2x^2}} = -\int_x \frac{d_x(x^{-1}+cx)}{x^{-1}+cx}\cdot\frac{1}{\sqrt{(x^{-1}+cx)^2+p-2c}}.$$

$$= \frac{1}{\sqrt{p-2c}}\log.\frac{x^{-1}+cx}{\sqrt{p-2c}+\sqrt{x^{-2}+p+c^2x^2}}.$$

$$\therefore \int_x \frac{p+qx^2+pc^2x^4}{1-c^2x^4}\cdot\frac{1}{\sqrt{1+px^2+cx^4}} =$$

$$\frac{2pc+q}{4c\sqrt{2c+p}}\log.\left\{\frac{1-cx^2}{x\sqrt{2c+p}+\sqrt{1+px^2+c^2x^4}}\right\} + \frac{2pc-q}{4c\sqrt{p-2c}}\log.\left\{\frac{1+cx^2}{x\sqrt{p-2c}+\sqrt{1+px^2+cx^4}}\right\}.$$

$$\int \frac{\sqrt{a+bx^2}}{x^4} = \int x^{-3}\sqrt{ax^{-2}+b}\cdot = \frac{2}{3}\frac{(ax^{-2}+b)^{\frac{3}{2}}}{-2a} = -\frac{1}{3a}\cdot(ax^{-2}+b)^{\frac{3}{2}}$$

$$= -\frac{1}{3a}\cdot\frac{\{a+bx^2\}^{\frac{3}{2}}}{x^3}\cdot$$

$$\int \frac{1}{(1+x)\sqrt{1-x^2}} = \int_x \frac{1}{(1+x)\sqrt{2\cdot(1+x)-(1+x)^2}} = \frac{1}{(1+x)^2\sqrt{\frac{2}{1+x}-1}}$$

$$= -\sqrt{\frac{2}{1+x}-1} \qquad = -\sqrt{\frac{1-x}{1+x}}\cdot$$

$$\int \frac{1}{x\sqrt{a+bx^n}} = \int_x \frac{x^{\frac{n}{2}-1}}{x^{\frac{n}{2}}\sqrt{a+bx^n}} = \frac{1}{\sqrt{b}}\int_{x^{\frac{n}{2}}} \frac{1}{x^{\frac{n}{2}}\sqrt{\frac{a}{b}+x^n}}\cdot$$

$$= \frac{1}{\sqrt{a}}\log\left\{\frac{x^{\frac{n}{2}}}{\left(\frac{a}{b}\right)^{\frac{1}{2}}+\sqrt{\frac{a}{b}+x^n}}\right\}\cdot$$

$$\frac{1+cx^2}{1-cx^2}\cdot\frac{1}{\sqrt{1+bx^2+c^2x^4}}\cdot = \int_x \frac{x^{-2}+c}{x^{-1}-cx}\cdot\frac{1}{\sqrt{x^{-2}+b+c^2x^2}}$$

$$= -\int_x \cdot\frac{d_x\cdot(x^{-1}-cx)}{x^{-1}-cx}\cdot\frac{1}{\sqrt{(x^{-1}-cx)^2+b+2c}}$$

$$= \frac{1}{\sqrt{b+2c}}\cdot\log_x\left\{\frac{x^{-1}-cx}{b+2c+\sqrt{x^{-2}+b+c^2x^2}}\right\}\cdot$$

$$\int \frac{1}{x\sqrt{a+bx+cx^2}} = \int x \frac{x^{-2}}{\sqrt{ax^{-2}+bx^{-1}+c}} = -\frac{1}{2\sqrt{a}} \cdot \int x \frac{-2ax^{-2}}{\sqrt{\left(ax^{-1}+\frac{b}{2}\right)^2+\frac{4ac-b^2}{4}}}$$

$$= -\frac{1}{2\sqrt{a}} \log\left(ax^{-1}+\frac{b}{2}\right) + x\sqrt{ax^{-1}+bx^{-1}+c}\Big)$$

$$\int x \frac{x}{\sqrt{a+bx+cx^2}} = \frac{1}{2c} \int x \frac{2cx+b}{\sqrt{a+bx+cx^2}} - \frac{b}{2c} \int x \frac{1}{\sqrt{a+bx+cx^2}}$$

$$\int x \frac{x^2}{\sqrt{a+bx+cx^2}} = \frac{1}{2c} \int x \cdot \frac{2cx+b}{\sqrt{a+bx+cx^2}} - \frac{b}{2c} \cdot \int x \frac{x}{\sqrt{a+bx+cx^2}}$$

$$= \frac{1}{2c} \cdot x\sqrt{a+bx+cx^2} - \frac{1}{c} \cdot \int x \sqrt{a+bx+cx^2}$$

$$- \frac{b}{2c^2} \sqrt{a+bx+cx^2} + \frac{b^2}{4c} \int x \frac{1}{\sqrt{a+bx+cx^2}}$$

$$\int x\sqrt{a^2+x^2}\cdot \log_e\left(\frac{x^2-b^2}{b^2}\right) = \frac{1}{3}(a^2+x^2)^{\frac{3}{2}}\cdot \log_e\left(\frac{x^2-b^2}{b^2}\right)$$

$$-\frac{2}{3}\int \frac{(a^2+x^2)^{\frac{3}{2}}x}{x^2-b^2}\,dx$$

$$\text{Now } \frac{(a^2+x^2)^{\frac{3}{2}}x}{x^2-b^2} = x\sqrt{a^2+x^2}\left\{\frac{a^2+x^2-b^2+b^2}{(x^2-b^2)}\right\}$$

$$= x\sqrt{a^2+x^2} + c^2\,\frac{x\sqrt{a^2+x^2}}{x^2-b^2}$$

$$x\sqrt{a^2+x^2}+c^2\left\{\frac{a^2x+x^3-b^2x+b^2x}{(x^2-b^2)\sqrt{a^2+x^2}}\right\}$$

$$= x\sqrt{a^2+x^2} + c^2\,\frac{x}{\sqrt{a^2+x^2}} + c^4\,\frac{x}{(x^2-b^2)\sqrt{a^2+x^2}}$$

$$\int x\sqrt{a^2+x^2} = \frac{1}{3}\cdot(a^2+x^2)^{\frac{3}{2}}$$

$$\int \frac{x}{\sqrt{a^2+x^2}} = c^2\sqrt{a^2+x^2}$$

$$\int \frac{x}{(x^2-b^2)\sqrt{a^2+x^2}} = c^4\int \frac{1}{z^2-(a^2+b^2)}$$ where $z=\sqrt{a^2+x^2}$

$$= \frac{c^4}{2\sqrt{a^2+b^2}}\cdot \log_e\left(\frac{z-c}{z+c}\right)$$

$$= \frac{c^3}{2}\log_e\left\{\frac{\sqrt{a^2+x^2}-c}{\sqrt{a^2+x^2}+c}\right\}$$

$$= \frac{c^3}{2}\cdot\log\left\{\frac{a^2+x^2-c^2}{\{\sqrt{a^2+x^2}+c\}^2}\right\} = \frac{c^3}{2}\cdot\log_e\frac{x^2-b^2}{(\sqrt{a^2+x^2}+c)^2}$$

$$= c^3\log_e\frac{\sqrt{x^2-b^2}}{\sqrt{a^2+x^2}+c}$$

$$\int x\sqrt{a^2+x^2}\log_e\left(\frac{x^2-b^2}{b^2}\right) =$$

$$\frac{1}{3}(a^2+x^2)^{\frac{3}{2}}\log_e\left(\frac{x^2-b^2}{b^2}\right) - \frac{2}{3}\left\{c^2\sqrt{a^2+x^2} + \frac{1}{3}(a^2+x^2)^{\frac{3}{2}} + c^3\log_e\left(\frac{\sqrt{x^2-b^2}}{\sqrt{a^2+x^2}+c}\right)\right\}$$

$$2\cos(mx^2+\alpha)\cos\gamma x = e^{(mx^2+\alpha)\sqrt{-1}} + e^{-(mx^2+\alpha)\sqrt{-1}} \cos\gamma x$$

$$= e^{\alpha\sqrt{-1}}\, e^{mx^2\sqrt{-1}} \cos\gamma x$$

$$\int_0^\infty e^{-a^2x^2}\cos\gamma x = \sqrt{\frac{\sqrt{\pi}}{2a}}\, e^{-\frac{\gamma^2}{4a^2}}$$

Let $-a^2 = m\sqrt{-1}$

$$\therefore e^{\alpha\sqrt{-1}} \int_0^\infty e^{m\sqrt{-1}\,x^2}\cos\gamma x = \frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{-m\sqrt{-1}}}\, e^{-\frac{\gamma^2}{4m\sqrt{-1}}} \times e^{\alpha\sqrt{-1}}$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{m}}\, \frac{e^{+\frac{\gamma^2}{4m}\sqrt{-1}}}{e^{-\frac{\pi}{4}\sqrt{-1}}}\, e^{\alpha\sqrt{-1}}$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{m}}\, e^{\left(\alpha+\frac{\pi}{4}+\frac{\gamma^2}{4m}\right)\sqrt{-1}}$$

$$\left.\begin{aligned} &e^{\alpha\sqrt{-1}} \int_0^\infty e^{-m\sqrt{-1}\,x^2}\cos\gamma x = \frac{1}{2}\sqrt{\frac{\pi}{m}} \cdot e^{-\left(\alpha+\frac{\pi}{4}+\frac{\gamma^2}{4m}\right)\sqrt{-1}} \\ &\therefore \int_0^\infty \cos mx^2 \cos\alpha\cos\beta x = \frac{1}{4}\sqrt{\frac{\pi}{m}} \left\{ e^{\left(\alpha+\frac{\pi}{4}+\frac{\gamma^2}{4m}\right)\sqrt{-1}} + e^{-\left(\alpha+\frac{\pi}{4}+\frac{\gamma^2}{4m}\right)} \right\} \end{aligned}\right\}$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{m}} \cdot \cos\left(\alpha+\frac{\pi}{4}-\frac{\gamma^2}{4m}\right)$$

$$\int_0^\infty \frac{\sin x \cos nx}{x} = \frac{1}{2} \int_0^\infty \left( \frac{\sin (n+1)x}{x} + \frac{\sin (1-n)x}{x} \right).$$

$$\int_0^\infty e^{-ax} \cos nx = \frac{a}{n^2 + a^2}.$$

$$\int_0^\infty e^{-ax} \frac{\sin nx}{x} = \tan^{-1} \frac{n}{a}.$$

Let $a = 0$

$$\therefore \int_0^\infty \frac{\sin nx}{x} = \frac{\pi}{2}.$$

$$\therefore \int_0^\infty \frac{\sin (n+1)x}{x} = \frac{\pi}{2}.$$

$$\int_0^\infty \frac{\sin (1-n)x}{x} = \frac{\pi}{2} \text{ if } n \text{ is less than } 1.$$

$$= -\frac{\pi}{2} \text{ if } n \text{ is greater than } 1.$$

$$\therefore \int_0^\infty \frac{\sin x \cos nx}{x} = \frac{1}{2} \left\{ \frac{\pi}{2} + \frac{\pi}{2} \right) \text{ or } \frac{1}{2} \left( \frac{\pi}{2} - \frac{\pi}{2} \right)$$

$$= \frac{\pi}{2} \text{ or zero, according as}$$

$n$ is less or greater than unity.

$$\int e^{ax} \sin(mx+\alpha) = \frac{1}{a} e^{ax} \sin(mx+\alpha) - \frac{m}{a}\int e^{ax} \cos(mx+\alpha).$$

$$= \frac{1}{a} e^{ax} \sin(mx+\alpha) - \frac{m}{a^2} e^{ax} \cos(mx+\alpha) - \frac{m^2}{a^2}\int e^{ax} \sin(mx+\alpha).$$

$$= \frac{e^{ax}}{m^2+a^2}\left\{ a \sin(mx+\alpha) - m \cos(mx+\alpha)\right\}.$$

$$\text{Let } \tan\theta = \frac{m}{a} \quad \therefore \sin\theta = \frac{m}{\sqrt{a^2+m^2}} \cdot \cos\theta = \frac{a}{\sqrt{a^2+m^2}}.$$

$$\therefore \int e^{ax} \sin(mx+\alpha) = \frac{e^{ax}}{\sqrt{a^2+m^2}} \sin(mx+\alpha-\theta).$$

$$= \frac{e^{ax}}{a \sec\theta} \sin(mx+\alpha-\theta)$$

$$\therefore \int_x^r e^{ax} \sin(mx+\alpha) = \frac{e^{ax}}{(a \sec\theta)/r} \sin(mx+\alpha-\theta).$$

$$\int x^n e^{ax} \sin(mx+\alpha) = x^n \int e^{ax} \sin(mx+\alpha) - n \int \left\{ x^{n-1} \int e^{ax} \sin(mx+\alpha)\right\}$$

$$= \frac{x^n \cdot e^{ax}}{a \sec\theta} \sin(mx+\alpha-\theta) - \frac{n}{a\sec\theta}\int x^{n-1} e^{ax} \sin(mx+\alpha-\theta).$$

$$\int x^{n-1} \cdot e^{ax} \sin(mx+\alpha-\theta) = \frac{x^{n-1} e^{ax}}{a \sec\theta} \sin(mx+\alpha-2\theta) - \frac{n(n-1)}{(a\sec\theta)^2} x^{n-2} \cdot e^{ax} \sin(mx+\alpha-2\theta)$$

$$\int x^n e^{ax} \sin(nx+a) = \frac{x^n e^{ax}}{a\sec\theta} \sin(nx+a-\theta)$$

$$- \frac{n x^{n-1} e^{ax}}{(a\sec\theta)^2} \sin(nx+a-2\theta)$$

$$+ \frac{n(n-1) e^{ax}}{(a\sec\theta)^3} x^{n-2} \sin(nx+a-3\theta) \&c.$$

The last term being $\frac{n(n-1)\ldots 2.1}{(a\sec\theta)^{n+1}} \sin(nx+a-(n+1)\theta)$.

$$\text{Cot } mx = \frac{1}{2}\left(e^{mx\sqrt{-1}} + e^{-mx\sqrt{-1}}\right)$$

$$\text{Cot } mx = \frac{1}{2}\left(e^{mx\sqrt{-1}} + e^{-mx\sqrt{-1}}\right).$$

$$= \frac{1}{2} e^{\sqrt{-1}} \left\{ e^{mx} + e^{-mx} \right\}.$$

$$\therefore e^{-ax^2} \left\{ e^{mx} + e^{-mx} \right\} = 2 e^{-ax^2} \cdot e^{\sqrt{-1}} \cdot \text{Cot } mx.$$

$$= 2 e^{-ax^2\sqrt{-1}} \text{ Cot } mx.$$

Bert. $\int_x^0 e^{-ax^2} \cos cx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \cdot e^{-\frac{c^2}{4a}}.$

$$\therefore \int_x^0 e^{-ax^2} \left\{ e^{mx} + e^{-mx} \right\} = \sqrt{\frac{\pi}{a}}\, e^{\frac{m^2}{4a}}.$$

$$\int_L^0 e^{-ax^2} \frac{1}{2}\left(e^{cx\sqrt{-1}} + e^{-cx\sqrt{-1}}\right) = \frac{1}{2}\sqrt{\frac{\pi}{a}}\, e^{-\frac{c^2}{4a}}$$

$$\text{let } c = m\sqrt{-1}$$

$$\therefore c^2 = -m^2$$

$$\int_L^0 e^{-ax^2} \frac{1}{2}\left(e^{-mxi} + e^{mxi}\right) = \frac{1}{2}\sqrt{\frac{\pi}{a}} \cdot e^{\frac{m^2}{4a}}$$

$$\int_0^\infty \left\{ e^{mx\sqrt{-1}} + e^{-mx\sqrt{-1}} \right\} e^{-ax^2} dx = 2 \int_0^\infty e^{-ax^2} \cos mx \, dx.$$

$$\int_0^\infty e^{-ax^2} \cos cx \, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \cdot e^{-\frac{c^2}{4a}}.$$

$$\int_0^\infty e^{-ax^2} \left\{ e^{mx\sqrt{-1}} + e^{-mx\sqrt{-1}} \right\} dx = 2 \int_0^\infty e^{-ax^2} \cos mx \, dx.$$

$$= \sqrt{\frac{\pi}{a}} \cdot e^{-\frac{m^2}{4a}}.$$

$1+x = \tan^2 z. \quad \therefore \quad 1+x = \sec^2 z.$

$$\frac{dx}{2} = 2.\tan z.\sec^2 z.$$

$$\frac{x^{m-1}}{1+x} = \frac{\tan^{2m-2} z}{\sec^2 z} \qquad \text{when } x=0 \;\; z=0.$$
$$x=\infty \;\; z=\frac{\pi}{2}.$$

$$\int_0^\infty \frac{x^{m-1}}{1+x} = \int_0^{\frac{\pi}{2}} \frac{x^{m-1}}{1+x}\, d_2 x.$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{\tan^{2m-2}}{\sec^2 z} \tan z \sec^2 z.$$

$$= 2\int_0^{\frac{\pi}{2}} \tan^{2m-1} z.$$

$$\text{But} \int_0^\infty \frac{x^{m-1}}{1+x} = \frac{\pi}{\sin m\pi}$$

$$\therefore \int_0^{\frac{\pi}{2}} \tan^{2m-1} z = \frac{\pi}{2\sin m\pi}.$$

$$\int t^{2n+1} e^{-at^2} = \int t^{2n} \cdot t e^{-at^2} = -\frac{1}{2a} e^{-at^2} \cdot t^{2n} + \frac{n}{a} \int t^{2n-1} \cdot e^{-at^2}$$

Now the integrated part of this expression vanishes when $t=0$ or when $t=\infty$.

$$\therefore \int_0^\infty t^{2n+1} e^{-at^2} = \frac{n}{a} \int_0^\infty t^{2n-1} e^{-at^2}$$

$$= \frac{n(n-1)}{a^2} \int_0^\infty t^{2n-3} e^{-at^2} \quad \&c.$$

$$= \frac{n.(n-1) - 2.1}{a^{n+1}} \int_0^\infty t e^{-at^2}$$

$$\int_0^\infty e^{-at^2} = \frac{1}{2} \sqrt{\frac{\pi}{a}}$$

$$\therefore \int_0^\infty t^{2n+1} e^{-at^2} = \frac{n(n-1) - 2.1}{a^{n+1}}$$

$$\int t e^{-at^2} = -\frac{1}{2} e^{-at^2} + C$$

when $t = \infty$ the integral vanishes

$$t = 0 \quad C = \frac{1}{2}$$

$$\therefore \int_0^\infty t e^{-at^2} = \frac{1}{2}$$

$$\therefore \int_0^\infty t^{2n+1} e^{-at^2} = \frac{n(n-1) - 2.1}{2 a^{n+1}}$$

$$\int t^{2n} e^{-at^2} = \int t^{2n-1} \cdot t e^{-at^2}$$

$$= -\frac{1}{2a} \cdot t^{2n-1} \cdot e^{-at^2} + \frac{2n-1}{2a} \cdot \int t^{2n-2} \cdot e^{-at^2}$$

$$\therefore \int_0^\infty t^{2n} e^{-at^2} = \frac{2n-1}{2a} \cdot \int t^{2n-2} \cdot e^{-at^2}$$

$$= \frac{(2n-1)(2n-3) \cdots 5 \cdot 3 \cdot 1}{(2a)^n} \cdot \int_0^\infty e^{-at^2}$$

$$\int_0^\infty e^{-at^2} = \frac{1}{2} \sqrt{\frac{\pi}{a}}$$

$$\therefore \int_0^\infty t^{2n} e^{-at^2} = \frac{(2n-1)(2n-3) \cdots 5 \cdot 3 \cdot 1}{(2a)^n} \cdot \frac{1}{2} \sqrt{\frac{\pi}{a}}.$$

$$\tan\theta = \frac{a\sin cx}{1+a\cos cx} \quad \therefore \cos\theta = \frac{1+a\cos cx}{\sqrt{1+2a\cos cx+a^2}}$$

$$= \frac{1+a\cos cx}{r}.$$

$$\text{or } r = \frac{1+a\cos cx}{\cos\theta}.$$

$$\therefore r^n\cos n\theta = 1 + na\cos cx + \frac{n(n-1)}{1\cdot 2}a^2\cos^2 cx + \&c \left\{1 - \frac{n(n-1)}{1\cdot 2}\tan^2\theta\right.$$

$$\text{But } \tan^2\theta = \frac{a^2\sin^2 cx}{(1+a\cos cx)^2}.$$

$$\therefore r^n\cos n\theta = 1 + na\cos cx + \frac{n(n-1)}{1\cdot 2}\cos 2cx + \&c.$$

$$\int_x^\infty \frac{dx}{1+x^2} = \tan^{-1}x + C = \frac{\pi}{2}.$$

$$\int_x^\infty \frac{\cos cx}{1+x^2} = \frac{\pi}{2}e^{-c}.$$

$$\int_x^\infty \frac{\cos 2cx}{1+x^2} = \frac{\pi}{2}e^{-2c} \quad \&c \&c.$$

$$\therefore \int_x^\infty \frac{r^n\cos n\theta}{1+x^2} = \frac{\pi}{2}\left\{1 + ne^{-c} + \frac{n(n-1)}{1\cdot 2}e^{-2c} + \&c.\right\}$$

$$= \frac{\pi}{2}\left(1+e^{-c}\right)^n.$$

$$\tan^{-1}\frac{m\sin cx}{1-m\cos cx} = \frac{m\sin cx}{1-m\cos cx} + \frac{1}{3}\left\{\frac{m\sin cx}{1-m\cos cx}\right\}^3 + \frac{1}{5}\left\{\frac{m\sin cx}{1-m\cos cx}\right\}^5 + \&c$$

$$= m\sin cx + \frac{m^2}{2}\left\{2\sin cx\cos cx\right\} + \frac{m^3}{3}\left\{3\sin cx\cos^2 cx - \sin^3 cx\right\}$$

$$+ \frac{m^4}{4}\left\{4\sin cx\cos^3 cx + 4\sin^3 cx\cos cx\right\} + \alpha c.$$

$$= m\sin cx + \frac{m^2}{2}\sin 2cx + \frac{m^3}{3}\sin 3cx + \frac{m^4}{4}\sin 4cx + \alpha c.$$

$$\therefore \frac{x}{1+x^2}\tan^{-1}\left(\frac{m\sin cx}{1-m\cos cx}\right) = \frac{mx\sin cx}{1+x^2} + \frac{m^2}{2}\cdot\frac{x\sin 2cx}{1+x^2} + \alpha c.$$

$$\int_0^\infty \frac{x\sin cx}{1+x^2} = \frac{\pi}{2}b^{-c}; \quad \int_0^\infty \frac{x\sin 2cx}{1+x^2} = \frac{\pi}{2}b^{-2c} \alpha c.$$

$$\therefore \int_0^\infty \frac{x}{1+x^2}\tan^{-1}\frac{m\sin cx}{1-m\cos cx} = \frac{\pi}{2}\left\{m\cdot b^{-c} + \frac{m^2}{2}b^{-2c} + \frac{m^3}{3}b^{-3c} + \alpha c\right\}$$

$$= \frac{\pi}{2}\log\left(\frac{1}{1-mb^{-c}}\right).$$

Diff.l Equations.

$$\frac{d^2y}{dx^2} + n^2 y - A\cos(nx+a) - B\cos(nx+\beta) = 0.$$

multiply by $\sin(nx+\beta)$.

$$\frac{d^2y}{dx^2}\sin(nx+\beta) + n^2 y\sin(nx+\beta) - A\cos(nx+a)\sin(nx+\beta) - B\cos(nx+\beta)\sin(nx+\beta) = 0.$$

$$\int \frac{d^2y}{dx^2}\sin(nx+\beta)\,dx = \int \sin(nx+\beta)\,dy - n\int dy\cos(nx+\beta)$$

$$= y\sin(nx+\beta) - ny\cos(nx+\beta) - \int n^2 y\sin(nx+\beta)\,dx$$

$$\therefore \int \frac{d^2y}{dx^2}\sin(nx+\beta)\,dx + \int n^2 y\sin(nx+\beta)\,dx = y\sin(nx+\beta) - ny\cos(nx+\beta).$$

$$\int \cos(nx+a)\sin(nx+\beta)\,dx = \frac{1}{n}\sin(nx+a)\sin(nx+\beta) - \frac{n}{n}\int \sin(nx+a)\cos(nx+\beta)\,dx$$

$$= \frac{1}{n}\sin(nx+a)\sin(nx+\beta) - \frac{n}{n}\left\{-\frac{1}{n}\cos(nx+a)\cos(nx+\beta) + \frac{n}{n}\int \cos(nx+a)\sin(nx+\beta)\,dx\right\}$$

$$= \frac{n\sin(nx+a)\sin(nx+\beta) + n\cos(nx+a)\cos(nx+\beta)}{n^2 - n^2}.$$

$\therefore$ the integral of the whole one is dividing throughout by $\sin^2(nx+\beta)$.

$$\frac{\frac{dy}{dx}}{\sin(nx+\beta)} - \frac{ny\cos(nx+\beta)}{\sin^2(nx+\beta)} = \frac{A}{n^2-n^2}\cdot\left\{\frac{-n\sin(nx+a)}{\sin(nx+\beta)} - \frac{n\cos(nx+a)\cos(nx+\beta)}{\sin^2(nx+\beta)}\right\}$$

$$-\frac{B}{4n}\frac{\cos(2nx+2\beta)}{\sin^2(nx+\beta)} + \frac{C}{\sin^2(nx+\beta)} = 0.$$

Integrate again and we have.

$$\frac{y}{\sin(nx+\beta)} - \frac{A}{n^2-n^2}\cdot\frac{\cos(nx+a)}{\sin(nx+\beta)} + \frac{B}{2n}\int 1\,dx - \left(C - \frac{B}{4n}\right)\int dx\,\frac{\cos(nx+\beta)}{n}$$

$$+ \frac{B}{2n}x + \left(\frac{B}{4n^2} - \frac{C}{n}\right)\cos(nx+\beta) + C_2.$$

$$\therefore y = \frac{A}{n^2-n^2}\cdot\cos(nx+a) - \frac{B}{2n}x\sin(nx+\beta) + \left(\frac{C}{n} - \frac{B}{4n^2}\right)\cos(nx+\beta)$$

$$- C_2\sin(nx+\beta)$$

$$= \frac{A}{n^2-n^2}\cos(nx+a) - \frac{B}{2n}x\sin(nx+\beta) + C_1\cos(nx+\beta_1).$$

(1) $\left(\frac{d^2}{d\theta^2} \mp n^2\right) u = 0.$

Let $u = \xi^m$ $\therefore$ by substitution $m^2 \mp n^2 = 0$

$\therefore m = n$ or $-n$.

$\therefore u = C\xi^{n\theta} + C_1 \xi^{-n\theta}$

(2). $\left(\frac{d^2}{d\theta^2} - n^2\right) u - a = 0.$

Let $u = u' - \frac{a}{n^2}$. or $u' = u + \frac{a}{n^2}$

$\therefore \frac{d^2}{d\theta^2} u' - n^2 u' = 0$

$\therefore u' = C\xi^{n\theta} + C_1 \xi^{-n\theta}$

or $u = -\frac{a}{n^2} \cdot C\xi^{n\theta} + C_1 \xi^{-n\theta}$.

(3). $\left(\frac{d^2}{d\theta^2} - n^2\right) u - \theta^n = 0.$

Let $u = u'$ be such a value of $u$ that $\left(\frac{d^2}{d\theta^2} - n^2\right) u = 0$

Let $u'$ be such a value

$\therefore u' = C\xi^{n\theta} + C_1 \xi^{-n\theta}.$

$\xi^{-n\theta} u' = C + C_1 \xi^{-2n\theta}$

$\frac{d}{d\theta} \cdot \xi^{-n\theta} u' = -2n \cdot \xi^{-2n\theta}.$

$\frac{1}{-2n} \xi^{2n\theta} \cdot \frac{d}{d\theta} \xi^{-n\theta} u' = 0.$ or $\xi^{-n\theta} \cdot \frac{d}{d\theta} \xi^{2n\theta} \cdot \frac{d}{d\theta} \xi^{-n\theta} u = \theta^n$

$\xi^{2n\theta} \cdot \frac{d}{d\theta} \cdot \xi^{-n\theta} u = \int \cdot \xi^{n\theta} \cdot \theta^n = \frac{1}{n} \xi^{n\theta} \cdot \theta^n - \int \xi^{n\theta} \cdot \theta^{n-1}$

$= \xi^{n\theta} \left\{ \frac{\theta^n}{n} - \frac{\theta^{n-1}}{n^2} + \frac{n-1}{n^2} \theta^{n-2} - \frac{(n-1)(n-2)}{n^3} \theta^{n-3} \cdots \right.$

$\left. \cdots (-1)^{n-1} \frac{(n-1)(n-2) \cdots 3 \cdot 2}{n^n} \cdot \theta + (-1)^n \frac{(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1}{n^{n+1}} \right\} + C$

$\therefore \xi^{-n\theta} u = \int \xi^{-n\theta} \left\{ \frac{\theta^n}{n} - \frac{\theta^{n-1}}{n^2} + \frac{n-1}{n^3} \theta^{n-2} \cdots \right.$

$\left. \cdots (-1)^{n-1} \frac{(n-1)(n-2) \cdots 3 \cdot 2}{n^n} \cdot \theta + (-1)^n \frac{(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1}{n^{n+1}} \right\} + C^{-2n\theta}$

see next page.

$$\int_\theta^{-n\theta} u = \int_\theta^{-n\theta} \left\{ \frac{\theta^n}{n} - \frac{\theta^{n-1}}{n^2} + \frac{n-1}{n^2}\theta^{n-2} + \cdots - (-1)^{\frac{(n-1)\cdot(n-2)\cdots 3\cdot 2}{n^n}} \cdot \theta + (-1)^n \frac{(n-1)\cdots 3\cdot 2\cdot 1}{n^{n+1}} \right\} + \int_\theta^{-n\theta} C$$

$$\text{Now } \int_\theta^{-n\theta} \frac{e^{-n\theta}}{n} \cdot \theta^n = -\frac{e^{-n\theta}}{n} \left\{ \frac{\theta^n}{n} + \frac{\theta^{n-1}}{n^2} + \frac{n-1}{n^2}\theta^{n-2} + \cdots \cdots (+1)^n \frac{(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{n^{n+1}} \right\}.$$

$$-\int_\theta^{-n\theta} \frac{e^{-n\theta}}{n^2} \theta^{n-1} = \frac{e^{-n\theta}}{n^2} \left\{ \frac{\theta^{n-1}}{n} + \frac{(n-1)\theta^{n-2}}{n^2} + \cdots (+1)^n \frac{(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{n^{n+1}} \right\}.$$

$$\therefore \frac{1}{n}\left\{ \int_\theta^{-n\theta} e^{-n\theta}\theta^n - \int_\theta^{-n\theta} e^{-n\theta}\cdot\theta^{n-1} \right\} = -\frac{1}{n^2} e^{-n\theta}\cdot\theta^n.$$

$$\text{Again } \frac{n-1}{n^2}\int_\theta^{-n\theta} e^{-n\theta}\cdot\theta^{n-2} = -\frac{n-1}{n^3} e^{-n\theta}\cdot\theta^{n-2} + \frac{(n-1)(n-2)}{n^3}\int_\theta^{-n\theta} e^{-n\theta}\cdot\theta^{n-3}.$$

$$\therefore \frac{n-1}{n^2}\int_\theta^{-n\theta} e^{-n\theta}\theta^{n-2} - \frac{(n-1)(n-2)}{n^3}\int_\theta^{-n\theta} e^{-n\theta}\theta^{n-3} = -\frac{n-1}{n^3} e^{-n\theta}\cdot\theta^{n-2}.$$

whereforc we have at last.

$$\int_\theta^{-n\theta} u = -\frac{1}{n^2} e^{-n\theta}\cdot\left\{ \frac{n\theta^n}{n^2} + \frac{n(n-1)\theta^{n-2}}{n^2} + \frac{n\cdot(n-1)(n-2)(n-3)}{n^4}\theta^{n-4} + \&c \right\} \beta + \alpha$$

Now in the series & the number of terms is $n+1$. if $n$ be odd. the last term is negative and there will be $\frac{1}{2}(n+1)$ pairs of terms. and $= {}^n B$ will become.

$$\int_\theta^{-n\theta} u = -\frac{1}{n^2} e^{-n\theta}\left\{ \frac{n\theta^n}{n} + \frac{n(n-1)\theta^{n-2}}{n^2} + \cdots (+1)\cdot \frac{n(n-1)\cdots 3\cdot 2}{n^{n-1}}\theta \right\} - \frac{1}{2n} C^{-2n\theta} + C_1$$

If $n$ be even becomes.

$$\int_\theta^{-n\theta} u = -\frac{1}{n^2} e^{-n\theta}\left\{ \frac{n\theta^n}{n} + \frac{n(n-1)\theta^{n-2}}{n^2} + \cdots (+1)\cdot \frac{n(n-1)\cdots 4\cdot 3}{n^{n-2}}\theta^2 + \frac{n(n-1)\cdots 3\cdot 2\cdot 1}{n^n} \right\}$$

$$\therefore u = -\frac{1}{n^2}\left\{ \frac{n\theta^n}{n} + \frac{n(n-1)\theta^{n-2}}{n^2} + \frac{n(n-1)(n-2)(n-3)\theta^{n-4}}{n^4} \cdots + \frac{n(n-1)(n-2)\cdots 3\cdot 2}{n^{n-1}}\theta \right\} - \frac{1}{2n} C^{-2n\theta} + C_1 e^{n\theta} + C_2 e^{-n\theta}.$$

$$\text{or} = -\frac{1}{n^2}\left\{ \frac{n\theta^n}{n} + \frac{n(n-1)\theta^{n-2}}{n^2} \cdots + \frac{n(n-1)\cdots 3\cdot 2\cdot 1}{n^n} \right\} + C_1 e^{n\theta} + C_2 e^{-n\theta}.$$ according as $n$ is odd or even.

The last example may be more conveniently performed thus.

$$\left(\frac{d^2}{d\theta^2} - n^2\right) u = \theta^n$$

$$+ n^2 \left\{ \frac{1}{n^2}\frac{d^2}{d\theta^2} - 1 \right\} u = \theta^n \quad \text{or} \quad u = -\frac{1}{n^2} \cdot \frac{\theta^n}{\left(1 - \frac{1}{n^2}\frac{d^2}{d\theta^2}\right)}$$

$$= -\frac{1}{n^2}\left\{ \theta^n + \frac{n(n-1)}{n^2}\theta^{n-2} + \frac{n.(n-1)(n-2)(n-3)}{n^4} + \&c. \right\} + C_1 e^{n\theta} + C_2 e^{-n\theta}.$$

which is the same result as before.

4). $\frac{d^2u}{d\theta} - n^2 u + A \cos n\theta + B$.

As in example 3 we have.

$$e^{n\theta} \frac{d}{d\theta} e^{-n\theta} u = A \int e^{n\theta} \cdot \cos n\theta + B. = -\frac{A}{n} e^{n\theta} \cos n\theta + \frac{An}{n} \int e^{n\theta} \sin n\theta + B$$

$$= -\frac{A}{n} e^{n\theta} \cos n\theta - \frac{A}{n^2} e^{n\theta} \sin n\theta + A \int e^{n\theta} \cos n\theta + B$$

$$= -\frac{A}{2n} e^{n\theta} \left\{ \cos n\theta + B + \sin n\theta + B \right\} + C_1$$

$$\therefore e^{-n\theta} u = -\frac{A}{2n} \int e^{-n\theta} \cdot \cos n\theta + B + \sin n\theta + B + C_1 \int e^{-2n\theta}.$$

$$\int e^{-n\theta} \cos n\theta + B = -\frac{1}{n} e^{-n\theta} \cos n\theta + B - \int e^{-n\theta} \cdot \sin n\theta + B$$

$$\therefore \int e^{-n\theta} \left( \cos n\theta + B + \sin n\theta + B \right) = -\frac{1}{n} e^{-n\theta} \cos n\theta + B. + C_2$$

$$e^{-n\theta} u = \frac{A}{2n^2} e^{-n\theta} \cdot \cos n\theta + B - \frac{C_1}{2n} e^{-2n\theta} + C_2.$$

$$u = \frac{A}{2n^2} \cdot \cos n\theta + B + C_2 e^{n\theta} + C_3 e^{-n\theta}.$$

The last example may be more shortly performed thus.

$$d_\theta^2\left(u - \frac{A}{2n^2}\cos.n\theta+b\right) - n^2\left(u - \frac{A}{2n^2}\cos.n\theta+b\right) = 0.$$

$$\therefore u - \frac{A}{2n^2}\cos.n\theta+b = C_1 \ell^{n\theta} + C_2 \ell^{-n\theta}.$$

(5). $d_\theta^2 u - n^2 u + A\sin.n\theta+b + B\cos.p\theta+c = 0.$

As before we have.

$$\ell^{2n\theta}\ell^{-n\theta} d_\theta \ell^{-n\theta} u = -A\int \ell^{n\theta}.\sin.n\theta+b - B\int \ell^{n\theta}\cos.p\theta+c$$

$$-A\int \ell^{n\theta}\sin.n\theta+b = -\frac{A}{n}.\ell^{n\theta}\sin.n\theta+b + \frac{Am}{n}\int \ell^{n\theta}\cos.n\theta+b.$$

$$+\frac{Am}{n}\left\{\frac{1}{n}\ell^{n\theta}\cos.n\theta+b + \frac{m}{n}\int \ell^{n\theta}\sin.n\theta+b\right)$$

$$= -A\left\{\frac{n\ell^{n\theta}\sin.n\theta+b - m\ell^{n\theta}\cos.n\theta+b.}{m^2+n^2}\right\} + C$$

$$-B\int \ell^{n\theta}\cos.p\theta+c = -B.\left\{\frac{n\ell^{n\theta}\cos.p\theta+c + p\ell^{n\theta}\sin.p\theta+c}{p^2+n^2}\right. + C.$$

$$\therefore \ell^{2n\theta}\ell^{-n\theta} d_\theta.\ell^{-n\theta} u = -\frac{A\ell^{n\theta}}{m^2+n^2}\left\{n\sin.n\theta+b - m\cos.n\theta+b\right\} - \frac{B\ell^{n\theta}}{p^2+n^2}\left\{n\cos.p\theta+c + p\sin.p\theta+c\right\}$$

$$+C.$$

$$\ell^{-n\theta} u = -\frac{A}{m^2+n^2}.\int\left\{\ell^{-n\theta}\sin.n\theta+b - m\ell^{-n\theta}\cos.n\theta+b\right\} - \frac{B}{p^2+n^2}.\int\left\{n\cos.p\theta+c + p\sin.p\theta+c\right\} + C.$$

$$= +\frac{A}{m^2+n^2}\ell^{-n\theta}\sin.n\theta+b + \frac{B}{p^2+n^2}.\ell^{-n\theta}.\cos.p\theta+c - \frac{1}{2n}C_1 + C_2.$$

$$\therefore u = \frac{A}{m^2+n^2}.\sin.n\theta+b + \frac{B}{p^2+n^2}.\cos.p\theta+c + C_2\ell^{n\theta} + C_3\ell^{-n\theta}.$$

$u = \theta^{n-1} \cos\theta \cdot \frac{du}{\theta} = (n-1)\theta^{n-2} \cdot \cos\theta + \theta^{n-1} \sin\theta.$

$\frac{d^2u}{\theta} = (n-1)(n-2)\theta^{n-3} \cos\theta - 2(n-1)\theta^{n-2} \sin\theta - \theta^{n-1} \cdot \cos\theta.$

$\frac{d^4u}{\theta} u = \cos\theta \cdot \theta^{n-1} + 4(n-1)\sin\theta \cdot \theta^{n-2} - 6\cos\theta(n-1)(n-2)\theta^{n-3} - 4\sin\theta(n-1)(n-2)(n-3)\theta^{n-4} + \&c$

$\mathcal{L}\mathcal{L} -$

$u = r^{n-2} \cos^2\theta = \frac{1}{2} \cdot \theta^{n-2} \cdot \cos 2\theta + 1$

$\therefore \frac{d^2}{\theta} \cdot \frac{1}{2} \cdot \theta^{n-2} \cdot (\cos 2\theta + 1) = \frac{1}{2}(n-2)(n-3)\theta^{n-4} +$

$\frac{1}{2}\left(-2^2 \cos 2\theta \cdot \theta^{n-2} - 2^2 \cdot \sin 2\theta \cdot (n-2)\theta^{n-3} + \cos 2\theta \cdot (n-2)(n-3)\theta^{n-4}\right.$

---

$\frac{1}{n^2} \cdot \theta^n - \frac{a}{n}\cos\theta + \frac{a}{n^3}\cos\theta - \frac{a}{n^3}\cos\theta.$ $\mathcal{L}_a$

$\theta^{n-1} + \frac{n(n-1)}{n^2}$

$\frac{1}{} - \frac{2a}{n^2}\sin\theta + \frac{4a}{n^4}\sin\theta$ $\mathcal{L}_c$ $+ \frac{a^2}{2}\cos^2\theta - \frac{a^2}{n^2}\cos 2\theta$ $\mathcal{L}_a$

$\theta^{n-2} + n\frac{(n-1)(n-2)}{n^3}$

$\frac{a\cos\theta}{n} - 6\frac{a\cos\theta}{n^2}$ $\mathcal{L}_2$ $- \frac{2a^2\sin 2\theta}{n}$ $\mathcal{L}_a$

$\theta^{n-3}$

6)

As an extension of Ex. 3 we may take the following.

$$d_\theta^2 u - n^2 u + (\theta - a\cos\theta)^n = 0.$$

$$\therefore (d_\theta^2 - n^2) u = -(\theta - a\cos\theta)^n$$

$$-n^2 \left(1 - \frac{1}{n^2} d_\theta^2\right) u = -(\theta - a\cos\theta)^n$$

$$u = +\frac{1}{n^2} \left\{ \theta^n - n\theta^{n-1} a\cos\theta + \frac{n(n-1)}{1\cdot 2} \theta^{n-2} a^2 \cos^2\theta + \text{&c} \right\}$$

$$\left(1 - \frac{1}{n^2} d_\theta^2\right)$$

$$= \frac{1}{n^2} \left\{ \theta^n + \frac{n(n-1)}{n^2} \theta^{n-2} + n\frac{(n-1)(n-2)(n-3)}{n^4} \theta^{n-4} + \text{&c.} \right.$$

$$- \frac{a}{n} \left\{ \theta^{n-1}\cos\theta + \frac{1}{n^2} d_\theta^2 \left( \theta^{n-1}\cos\theta \right\} + \frac{1}{n^4} d_\theta^4 \left\{ \theta^{n-1}\cos\theta \right\} \text{&c.} \right.$$

$$+ \frac{(n-1)a^2}{1\cdot 2\cdot n} \cdot \left\{ \theta^{n-2}\cos^2\theta + \frac{1}{n^2} \cdot d_\theta^2 \left\{ \theta^{n-2}\cos^2\theta \right\} + \frac{1}{n^4} d_\theta^4 \left\{ \theta^{n-2}\cos^2\theta \right\} + \text{&c.} \right.$$

$$\cdot + \text{&c} \cdot \text{&c} \cdot \text{&c.}$$

$$= \frac{1}{n^2} \left\{ \theta^n + \frac{n(n-1)}{n^2} \cdot \theta^{n-2} + \frac{n \cdots (n-3)}{n^4} \theta^{n-4} \cdot + \text{&c.} \right.$$

$$- \frac{na}{n^2} \left\{ \cos\theta \cdot \theta^{n-1} - \frac{2}{n^2}\cos\theta \cdot \theta^{n-1} - \frac{2(n-1)}{n^2} \sin\theta\, \theta^{n-2} + \frac{(n-1)(n-2)\cos\theta}{n^2} \cdot \theta^{n-3} \right.$$

$$\left. + \frac{1}{n^4} \cos\theta \cdot \theta^{n-1} + 4\frac{(n-1)}{n^4} \sin\theta \cdot \theta^{n-2} - 6\frac{(n-1)(n-2)}{n^4} \cos\theta \cdot \theta^{n-3} - 4 \cdot \frac{n \cdots n-3}{n^4} \theta^{n-4} + \text{&c.} \right.$$

$$\frac{(n-1)a^2}{1\cdot 2\cdot n^2} \left\{ \cdots \cos^2\theta \cdot \theta^{n-2} \cdots \right.$$

$$\cdots - \frac{2}{n^2}\cos 2\theta \cdot \theta^{n-2} - \frac{2(n-2)}{n^2} \sin 2\theta\, \theta^{n-3} + \frac{(n-2)(n-3)}{2n^2} \cos 2\theta \cdot \theta^{n-4} + \text{&c.} \right.$$

$$\text{&c} \sim \sim \sim$$

$$\cdots + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\, n^2} \cdot a^4 \cos^4\theta \cdot \theta^{n-4} \cdot$$

Collect the results and reduce.

see opposite.

$d_x^2 y - n^2 y - A \cos nx + a = B \cos nx + \beta = 0$.

$p = d_x y = f(x, y)$ the proposed diff. eq.

$y = \phi(x, c)$ the complete integral (1)

Suppose $y = u$ be a solution to suppose it possible proceeding by giving $c$ the value $c'$

$\therefore \quad \phi(x, c) - u =$ a series of ascending powers of $c - c'$

$= az$ suppose where $a$ is the lowest power of $c - c'$ & $z$ a function which does not vanish when $c = c'$

$\therefore \quad y = u + az$ is the complete value of $y$

$\therefore d_x y = d_x u + a d_x z = f(x, u + az)$

$= f(x, u) + az \cdot d_{y=u} f(x, u) + \ldots$

but $\because$ $u$ is a solution

$\therefore d_x u = f(x, u)$

$\therefore a d_x z = az \cdot d_{y=u} f(x, u) + \ldots$ (2)

Now if $y = u$ be a particular integral obtainable by making $c = c'$ i.e. $a = 0$ & $\therefore$ (2) is satisfied but if $y = u$ be a singular solution then (2) must be satisfied by the vanishing of $z$ independent of the value of $a$ i.e.

but $\therefore$ $d_y z$ does not necessarily vanish when $z = 0$

as then $d_y z (= v) = \frac{\text{finite}}{0} = \infty$ . when $y = 0$

which cond$^n$ is known as $d_{y,1} T = 0$ whilst

$d_{y,1} T$ remains finite $\therefore$ $d_y T = - \frac{d_{y,1} T}{d_{y,1} T}$

when $T = 0 =$ $\phi(x, y, p)$ is the form of the Equation,

then solutions deduced by this condition are singular solutions, however.

$$d_y T = 0 = d_{x,1} T + d_{p,1} T \cdot d_y p =$$

$$\therefore d_y p = - \frac{d_{x,1} T}{d_{p,1} T}$$

satisfy $p$ between by an $= \chi$ $\phi(x, y, p) = 0 = T$ suppose

then $d_x T = 0 = d_{x,1} T + d_{p,1} T \cdot d_y p$

$$\therefore d_y p = - \frac{d_{x,1} T}{d_{p,1} T}$$

$\therefore$ if the cond$^n$ $d_y p = \infty$ is equivalent to

$d_{p,1} T = 0$ whilst at same time $d_{x,1} T$ is finite

and the solutions deduced from this condition are

singular solutions, but if $y = a$ do not satisfy

both these cond$^{ns}$ it is a particular integral

$$\frac{x^{p-1}}{(1+ax)(1-x)^p} = \frac{x^{p-1}}{(1-x)^p}\left\{1 - ax + a^2x^2 - \&c\right\}.$$

$$\text{Prop.} \int_x^{0\,/\,1} \frac{x^{p-1}}{(1-x)^p} = \frac{\pi}{\sin p\pi}.$$

$$\text{Cor.} \quad -a \int_x^{0\,/\,1} \frac{x^p}{(1-x)^p}$$

$$\int_x^{0\,/\,1} \frac{x^{m+p-1}}{(1-x)^p} = \frac{m+p-1}{m} \int_x^{0\,/\,1} \frac{x^{m+p-2}}{(1-x)^p}$$

$$= \frac{m+p-1}{m} \cdot \frac{m+p-2}{m-1} \cdots \frac{p}{1} \int_x^{0\,/\,1} \frac{x^{p-1}}{(1-x)^p}$$

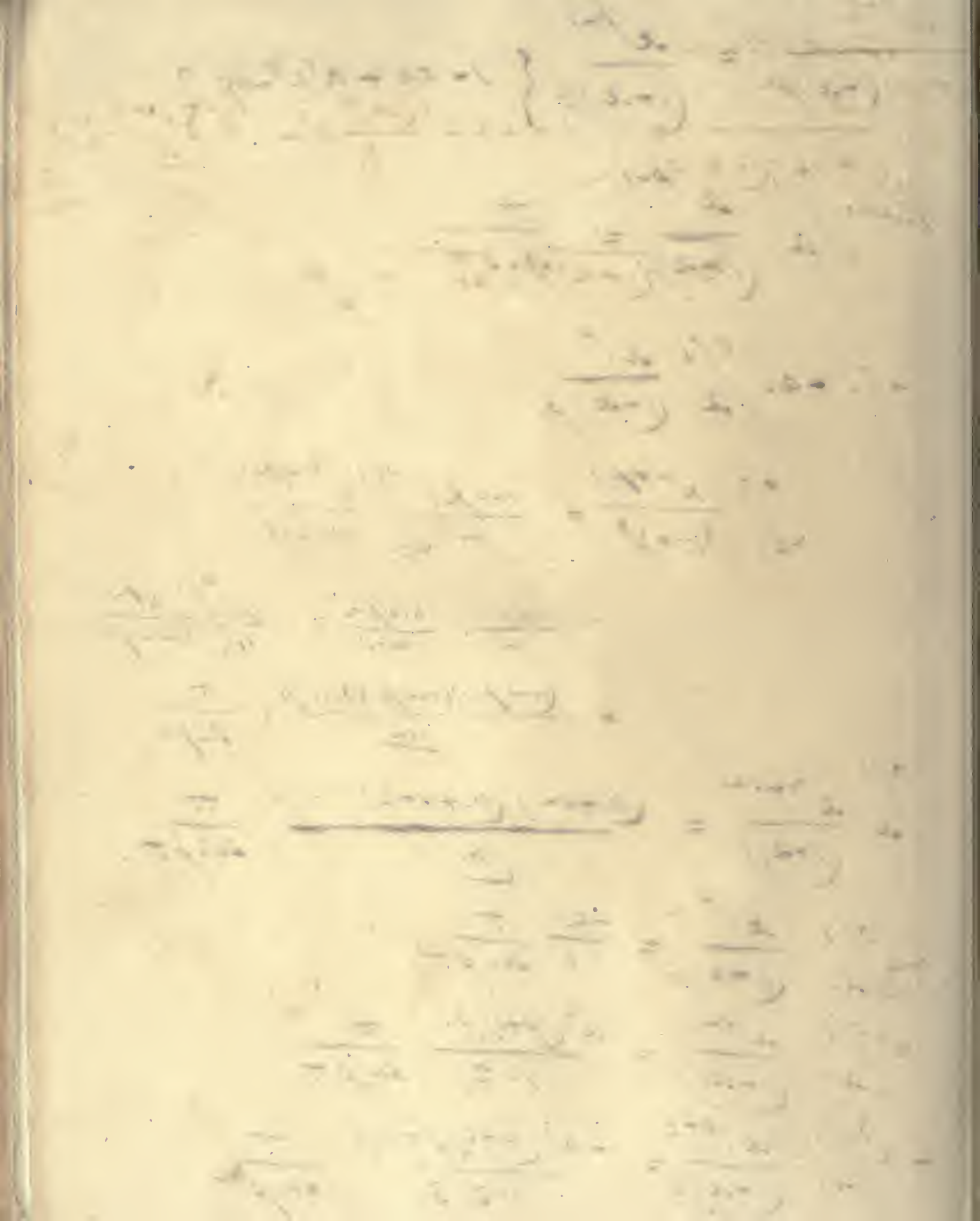
$$= \frac{(m+p-1)(m+p-2)\cdots p}{\lfloor m} \cdot \frac{\pi}{\sin p\pi}$$

$$\int_x^{0\,/\,1} \frac{x^{m+p-1}}{(1-x)^p} = \frac{(m+p-1)(m+p-2)\cdots p}{\lfloor m} \cdot \frac{\pi}{\sin p\pi}.$$

$$-a \int_x^{0\,/\,1} \frac{x^p}{(1-x)^p} = \frac{-ap}{1} \frac{\pi}{\sin p\pi}.$$

$$a^2 \int_x^{0\,/\,1} \frac{x^{p+1}}{(1-x)^p} = \frac{a^2(p+1)p}{1\cdot 2} \frac{\pi}{\sin p\pi}.$$

$$-a^3 \int_x^{0\,/\,1} \frac{x^{p+2}}{(1-x)^p} = \frac{-a^3(p+2)(p+1)p}{1\cdot 2\cdot 3} \frac{\pi}{\sin p\pi}.$$



$\therefore$ By addition

$$\int_0^1 \frac{x^{p-1}}{(1+ax)(1-x)^p} = \left(1 - pa + \frac{p(p+1)}{1\cdot 2}a^2 - \frac{p(p+1)(p+2)}{\underline{|3}}a^3 \cdots\right)\frac{\pi}{\sin p\pi}$$

$$= \frac{1}{(1+a)^p}\,\frac{\pi}{\sin p\pi}.$$

$$(1-x^2)\frac{dy}{dx} + 2xy = ax^2 + b.$$

$$\frac{dy}{dx} + \frac{2x}{1-x^2}\cdot y = ax^2 + b.$$

Integrating factor is $e^{\int \frac{2x}{1-x^2}}$

$$= \frac{1}{1-x^2}.$$

$$\therefore \int \left\{ \frac{y}{(1-x^2)} \right\} = \frac{ax^2 + b}{(1-x^2)^{1/2}}$$

$$= -\frac{a}{1-x^2} + \frac{a+b}{(1-x^2)^{1/2}}.$$

$$\therefore \frac{y}{1-x^2} = -\frac{a}{2}\cdot \log_e \left\{ \frac{1+x}{1-x} \right\} + (a+b)\left\{ \frac{x}{(1-x^2)^{1/2}} + \frac{1}{2}\int \frac{1}{(1-x^2)^{1/2}} \right.$$

$$= -\frac{a}{2}\cdot \log_e \left( \frac{1+x}{1-x} \right) + \{a+b\}\frac{x}{(1-x^2)^{1/2}} + \frac{1}{4}\log_e \left( \frac{1+x}{1-x} \right)$$

$$= \frac{b-a}{4}\cdot \log_e \left( \frac{1+x}{1-x} \right) + \frac{a+b}{2}\cdot \frac{x}{(1-x^2)}$$

$$(1+x^2)\frac{dy}{dx} + xy + 1 = 0.$$

$$\frac{dy}{dx} + \frac{x}{1+x^2}y + \frac{1}{1+x^2} = 0.$$

Integrating Factor $= \sqrt{1+x^2}$.

$$\sqrt{1+x^2}\cdot\frac{dy}{dx} + \frac{xy}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}} = 0.$$

$$d_x\left\{y\sqrt{1+x^2}\right) + \frac{1}{\sqrt{1+x^2}} = 0.$$

$$y\sqrt{1+x^2} = \log c - \log.\left(x+\sqrt{1-x^2}\right)$$

$$= \log.\left(\frac{c}{x+\sqrt{1-x^2}}\right).$$

$$\therefore \quad c = \left(x+\sqrt{1-x^2}\right)\mathcal{E}^{y\sqrt{1+x^2}}.$$

$$\frac{dy}{x} - \frac{2}{\sqrt{x}} y = a\frac{\sqrt{y}}{\sqrt{x}}.$$

$$\frac{dy_x}{2\sqrt{y}} - \frac{1}{\sqrt{x}}\sqrt{y} = \frac{a}{2\sqrt{x}}$$

$$d_x\sqrt{y} - \frac{1}{\sqrt{x}}\cdot\sqrt{y} = \frac{a}{2\sqrt{x}}.$$

Integrating factor $e^{-2\sqrt{x}}$.

$$\therefore d_x\left(e^{-2\sqrt{x}}\cdot\sqrt{y}\right) = a\cdot e^{-2\sqrt{x}}\frac{}{2\sqrt{x}}.$$

$$\therefore \sqrt{y} = -\frac{a}{2}e^{2\sqrt{x}}\cdot\int x - \frac{e^{-2\sqrt{x}}}{2\sqrt{x}}.$$

$$= -\frac{a}{2}e^{2\sqrt{x}}\left\{e^{-2\sqrt{x}} + C_1\right\}.$$

$$= -\frac{a}{2}\left\{1 + C_1 e^{2\sqrt{x}}\right\}.$$

$$\frac{dz}{x} = \frac{a}{\sqrt{a^2 - y^2 - z^2}}$$

$$\therefore z = a\left\{\sin^{-1}\frac{x}{\sqrt{a^2 - y^2}} - \sin^{-1}\varphi(y)\right\}$$

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$$lx + my + nz = 0.$$

$$P = x. \quad Q = y. \quad R = -z.$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

$$\frac{dx}{x} = \frac{dy}{y} \quad \therefore \log\left(\frac{x}{y}\right) = \log c.$$

$$\frac{x}{y} = c$$

$$\frac{dx}{x} = -\frac{dz}{z}.$$

$$\therefore \log x + \log z = \log c_2.$$

$$\frac{x}{z} = c_2. \qquad xz = f\left(\frac{y}{x}\right)$$

$$\therefore \frac{x}{y} = f\left(\frac{x}{z}\right).$$

$y = bx$
$z = \dots$

$xy = ax = f(bx) = \frac{a}{b^2}\cdot(bx)^2$

$xz = \frac{a}{b}\cdot\left(\frac{y}{x}\right)^2.$

when $y = ax^2 \quad z = bx.$

$$\therefore \frac{1}{ax} = f\left(\frac{x}{bx}\right) = f\left(\frac{1}{b}\right).$$

$$d_x^2 z - d_y^2 z = ax + by.$$

$$\left(d_x^2 - d_y^2\right) z = ax + by.$$

$$\therefore z = \left(d_x^2 - d_y^2\right)^{-1} \cdot (ax + by)$$

$$= \left(d_x^2\right)^{-1} \cdot (ax + by) + \varphi(y + x) + \psi(y - x)$$

$$= \int_x^2 \cdot (ax + by) + \varphi(y + x) + \psi\, y - x$$

$$= \frac{ax^3}{6} + \frac{bx^2 y}{2} + \varphi \cdot \overline{y + x} + \psi\ \overline{y - x}$$

$x^2p + y^2q = z^2$.

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}.$$

$$\therefore -\frac{1}{x} + \frac{1}{y} = c.$$

$$-\frac{1}{x} + \frac{1}{z} = c_2.$$

$$\therefore \frac{1}{z} - \frac{1}{x} = f.\left(\frac{1}{y} - \frac{1}{x}\right)$$

$p\cdot(z+y)+q\cdot(z+x)=(x+y)$.

$$\frac{dx}{z+y}=\frac{dy}{z+x}=\frac{dz}{x+y}.$$

$$\therefore \frac{dx+dy+dz}{2(x+y+z)}=\frac{dx-dy}{x-y}=-\frac{dx-dz}{x-z}.$$

$$=-\frac{dy-dz}{y-z}$$

$$\therefore \frac{1}{2}\log.(x+y+z)+\log.(y-z)=\log c_1$$

$$(x+y+z)(y-z)^2=c_1$$

$$(x+y+z)(x-z)^2=c_2.$$

$$\therefore f\{(x+y+z)(x-z)^2,(x+y+z)(y-z)^2\}=0.$$

$$d_x^2 y + \frac{1}{x} \cdot d_x y - \frac{1}{x^2} \cdot y = 1$$

$$x \cdot d_x^2 y + d_x y - \frac{y}{x} = x \cdot$$

$$d_x^2 (xy) - d_x y - \frac{y}{x} = x \cdot$$

$$d_x xy =$$

$$\frac{y}{x} = z \cdot$$

$$y = xz \cdot$$

$$\therefore d_x y = z + x d_x z \cdot$$

$$d_x^2 y = 2 d_x z + x d_x^2 z \cdot$$

$$\log x = t \quad x = e^t \cdot \quad d_t x = e^t = x \cdot$$

$$d_x y = d_t y \, d_x t \cdot = \frac{d_t y}{d_t x} \cdot$$

$$= \frac{1}{x} \cdot d_t y$$

$$d_x^2 y = d_x \left( \frac{d_t y}{x} \cdot d_x t \right) = d_t \left( \frac{d_t y}{x} \cdot d_x t \right) d_x t \cdot$$

$$= \frac{d_t^2 y}{x} (d_x t)^2 + \frac{d_t y}{x} \cdot d_x^2 t$$

$$d_x y = \frac{d_t y}{d_t x} \cdot \quad d_x y = \frac{d_t x \, d_t y - d_t y \, d_t x}{(d_t x)^3} \cdot$$

$$\frac{dz}{t} + \frac{dx}{t} = 2\sqrt{ga}.$$

Also $(z-x)^2 + (a-y)^2 = a^2$.

$$\left(\frac{dz}{t} - \frac{dx}{t}\right)(z-x) = (a-y)\frac{dy}{t}.$$

$$2.\left(\sqrt{ga} - \frac{dx}{t}\right)(z-x) = (a-y)\frac{dy}{t}.$$

$$\text{or } \frac{dx}{t} = \sqrt{ga} - \frac{(a-y)\,dy}{2(z-x)}$$

$$\frac{dz}{t} = \sqrt{ga} + \frac{(a-y)\,dy}{2(z-x)}.$$

$$\therefore 2ga + \left(\frac{a-y}{z-x}\right)^2\left(\frac{dy}{t}\right)^2 + \left(\frac{dy}{t}\right)^2 = 2g(2a-y)$$

Let $\frac{dy}{t} = 0$ Then

$$x = 2a - y \quad \therefore y = a.$$

$y^2 - 2xyp + (1+x^2)p^2 - 1 = 0 = V.$

for a singular solution $\frac{dV}{dp} = 0$.

$\therefore -2xy + 2(1+x^2)p = 0.$

$$p = \frac{xy}{1+x^2}$$

$\therefore$ The curves.

$$y^2 - \frac{2x^2y^2}{(1+x^2)} + \frac{x^2y^2}{1+x^2} - 1 = 0.$$

$$y^2 - \frac{x^2y^2}{1+x^2} - 1 = 0.$$

$$y^2 - 1 - x^2 = 0 \qquad y^2 = 1 + x^2.$$

$$p = \frac{y - p^2}{x - p} \quad \text{(Find a dmyalon solution.)}$$

$$\therefore \frac{px - p^2 + p - y}{x - p}$$

$$px - p^2 + p - y = 0$$

$$x - 2p + 1 = 0$$

$$p = \frac{x+1}{2}$$

$$2\,\frac{x^2 + 2x}{4} - \frac{x^2 + 2x + 1}{4} + \frac{2x + 2}{4} - y = 0$$

$$\frac{x^2}{4} + \frac{2x}{4} + \frac{1}{4} - y = 0$$

$$\frac{1}{4}(x+1)^2 - y = 0$$

$$\left(y - x\cdot\frac{dy}{dx}\right)^2 + \left(x - \frac{y}{\frac{dy}{dx}}\right)^2 = a^2.$$

$$y^2 - 2xy\cdot p + x^2p^2 + x^2 - \frac{2xy}{p} + \frac{y^2}{p^2} = a^2.$$

$$\left(\frac{y^2}{p^2} + x^2 - \frac{2xy}{p}\right)(p^2+1) = a^2 \quad \therefore \left(\frac{y}{p} - x\right) = \frac{a}{\sqrt{p^2+1}}$$

$$\therefore \frac{y}{p^2} = \frac{ap}{(p^2+1)^{\frac{3}{2}}}.$$

$$y^2\cdot(p^2+1)^3 = a^2p^6.$$

$$y^{\frac{2}{3}}\cdot(p^2+1) = a^{\frac{2}{3}}p^2 \quad \therefore \left(y^{\frac{2}{3}} - a^{\frac{2}{3}}\right)p^2 = -y^{\frac{2}{3}}.$$

$$p^2 = \frac{-y^{\frac{2}{3}}}{y^{\frac{2}{3}} - a^{\frac{2}{3}}} = \frac{y^{\frac{2}{3}}}{a^{\frac{2}{3}} - y^{\frac{2}{3}}}$$

$$p^2+1 = \frac{-x^{\frac{2}{3}}}{y^{\frac{2}{3}} - a^{\frac{2}{3}}} = \frac{a^{\frac{2}{3}}}{a^{\frac{2}{3}} - y^{\frac{2}{3}}}.$$

$$\text{Put } \frac{y^2}{p^2} + x^2 - \frac{2xy}{p} = \frac{a^2}{p^2+1}$$

$$y^{\frac{4}{3}}\left(a^{\frac{2}{3}} - y^{\frac{2}{3}}\right) + x^2 - 2x\cdot y^{\frac{2}{3}}\left(a^{\frac{2}{3}} - y^{\frac{2}{3}}\right)^{\frac{1}{2}} = a^{\frac{4}{3}}\cdot\left(a^{\frac{2}{3}} - y^{\frac{2}{3}}\right).$$

$$-y^{\frac{2}{3}}\cdot\left(a^{\frac{2}{3}} - y^{\frac{2}{3}}\right)^{\frac{1}{2}} + x = a^{\frac{2}{3}}\cdot\left(a^{\frac{2}{3}} - y^{\frac{2}{3}}\right)^{\frac{1}{2}}.$$

$$x = \left(a^{\frac{2}{3}} - y^{\frac{2}{3}}\right)^{\frac{3}{2}}.$$

$$x^{\frac{2}{3}} = a^{\frac{2}{3}} - y^{\frac{2}{3}}.$$

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

$$y \cdot d_x^2 y + \sqrt{1+(d_x y)^2} = 0.$$

$$\frac{d_x^2 y}{\sqrt{1+(d_x y)^2}} + \frac{1}{y} = 0.$$

$$\frac{dp_x}{\sqrt{1+p^2}} + \frac{1}{y} = 0.$$

$$\frac{p \cdot dp_y}{\sqrt{1+p^2}} + \frac{1}{y} = 0.$$

$$\sqrt{1+p^2} + \log y = \log c$$

$$\sqrt{1+p^2} = \log \cdot \frac{c}{y}.$$

$$y \cdot d_x^2 y + \left(d_x y\right)^2 = 0$$

$$d_x \cdot \left(y \cdot d_x y\right) = 0.$$

$$\therefore y \cdot d_x y = C$$

$$d_x y = \frac{C}{y}.$$

$$dx = \frac{y}{C}.$$

$$\therefore x = \frac{1}{2}\frac{y^2}{C} + C'.$$

$$\frac{dy}{dx} + \frac{a}{1-x^2} \cdot y = \frac{1+x}{(1-x)^3}$$

$$\int_x P = \frac{a}{2} \log \frac{1+x}{1-x} = \log \left(\frac{1+x}{1-x}\right)^{\frac{a}{2}}$$

$$\therefore d_x \left( y \cdot \left(\frac{1+x}{1-x}\right)^{\frac{a}{2}} \right) = \int \frac{(1+x)^{\frac{a}{2}+1}}{(1-x)^{\frac{a}{2}+3}} \cdot dx$$

$$y = \left(\frac{1-x}{1+x}\right)^{\frac{a}{2}} \cdot \int_x \left(\frac{1+x}{1-x}\right)^{\frac{a}{2}} \cdot \frac{1+x}{(1-x)^3}$$

$$= \left(\frac{1-x}{1+x}\right)^{\frac{a}{2}} \cdot \int_x \left(\frac{1+x}{1-x}\right)^{\frac{a+2}{2}} \cdot \frac{1}{(1-x)^2}$$

$$= \left(\frac{1-x}{1+x}\right)^{\frac{a}{2}} \left\{ \frac{1}{2} \cdot \int_x \left(\frac{1+x}{1-x}\right)^{\frac{a+2}{2}} \cdot d_x \left(\frac{1+x}{1-x}\right) \right\}$$

$$= \left(\frac{1-x}{1+x}\right)^{\frac{a}{2}} \left\{ \frac{1}{a+4} \cdot \left(\frac{1+x}{1-x}\right)^{\frac{a+4}{2}} + C \right.$$

$$= \frac{1}{a+4} \cdot \left(\frac{1+x}{1-x}\right) + C \cdot \left(\frac{1-x}{1+x}\right)^{\frac{a}{2}}.$$

$$(1-x^2)^{\frac{1}{2}} \frac{dy}{dx} - ny = x(1-x^2)^{\frac{1}{2}}.$$

$$\frac{dy}{dx} - \frac{n}{(1-x^2)^{\frac{1}{2}}} y = x.$$

Integrating factor $= e^{\int \frac{-n}{(1-x^2)^{\frac{1}{2}}} dx} = e^{-n \sin^{-1} x}.$

$$\therefore \frac{d}{dx}\{y e^{-n \sin^{-1} x}\} = x e^{-n \sin^{-1} x}.$$

$$y = e^{n \sin^{-1} x} \int x e^{-n \sin^{-1} x} dx.$$

$$\int x e^{-n \sin^{-1} x} dx = -\frac{x\sqrt{1-x^2}}{n} e^{-n \sin^{-1} x} + \frac{1}{n} \int \sqrt{1-x^2}\, e^{-n \sin^{-1} x} dx.$$

$$= -\frac{x\sqrt{1-x^2}}{n} e^{-n \sin^{-1} x} - \frac{1}{n^2}(1-x^2) e^{-n \sin^{-1} x} - \frac{2}{n^2} \int \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} x e^{-n \sin^{-1} x} dx.$$

$$= -\left\{\frac{nx\sqrt{1-x^2} + 1 - x^2}{n^2+2}\right\} e^{-n \sin^{-1} x} + C$$

$$\therefore y = -\left\{\frac{nx\sqrt{1-x^2} + 1 - x^2}{n^2+2}\right\} + C e^{n \sin^{-1} x}.$$

$$dy - \frac{n}{\sqrt{1-x^2}} \cdot y = x$$

$$I = -n \sin^{-1} x$$

$$d(y \cdot e^{-n \sin^{-1} x}) = x \cdot e^{-n \sin^{-1} x}$$

$$y \cdot e^{-n \sin^{-1} x} = \frac{x^2}{2} \cdot e^{-n \sin^{-1} x} + \frac{n}{2} \int \frac{x^2}{\sqrt{1-x^2}}$$

$$= \frac{-x\sqrt{1-x^2}}{n} \cdot e^{-n \sin^{-1} x} + \frac{1}{n} \int e^{-n \sin^{-1} x} \cdot \sqrt{1-x^2}$$

$$= x\sqrt{1-x^2} \cdot \left(\frac{1}{\sqrt{1-x^2}} \cdot e^{-n \sin^{-1} x}\right)$$

$$= \frac{x^2}{2} \cdot e^{-n \sin^{-1} x} + \frac{n}{2} \int \frac{x^2}{\sqrt{1-x^2}} \cdot e^{\pm n \sin^{-1} x}$$

$$= \frac{x^2}{2} \cdot e^{-\phi} + \frac{n}{2} \int \frac{1}{\sqrt{1-x^2}} \cdot e^{-\phi} - \sqrt{1-x^2}\, e^{-\phi}$$

$$= \frac{x^2}{2} \cdot e^{-\phi} + \frac{n}{2}\left(-\frac{1}{n} e^{-\phi}\right) - \frac{n}{2} \cdot 1-x^2$$

$$\sin^{-1} x = \theta$$

$$x = \sin\theta$$

$$dx = \cos\theta$$

$$y \cdot e^{-n\theta} = \int x \cdot e^{-n\theta} = \int \sin\theta \cos\theta \cdot e^{-n\theta}$$

$$= \frac{1}{2} \cdot \int \sin 2\theta \cdot e^{-n\theta}$$

$$\frac{1}{2} \cdot e^{-n\theta} \cdot \frac{n \sin 2\theta + 2\cos 2\theta}{n^2+4}$$

$$= \frac{1}{2} \cdot e^{-n\theta} \cdot \frac{2 \cdot 1-2x^2 + 2n \cdot x\sqrt{1-x^2}}{n^2+4}$$

$$= -e^{-n\theta} \cdot \frac{n x\sqrt{1-x^2} + 1-2x^2}{n^2+4}$$

$$\frac{d^2y}{dx^2} = f(y).$$

$$2 \cdot \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} = 2 \cdot f(y) \cdot \frac{dy}{dx}.$$

$$\left(\frac{dy}{dx}\right)^2 = \phi(y) + C$$

$$\frac{dy}{dx} = \sqrt{\phi(y) + C}.$$

$$dx = \frac{1}{\sqrt{\phi(y) + C}}$$

$$x = \int \frac{1}{\sqrt{\phi(y) + C}}$$

$$E.\left(d^2_x y - \frac{dy}{x^1}.\, y\right) = 0.$$

$$d^2_x y = \frac{dp}{x} = \frac{dp}{y}.\frac{dy}{x} = p.\frac{dp}{y}.$$

$\therefore$ $E.\left(d^2_x y.\frac{dy}{x}.y\right)$ becomes.

$E.\left(p.\frac{dp}{y}.p.y\right)$ which can be reduced to an = n of the first order.

$$\frac{d^2y}{dx^2} + n^2 y = 0.$$

$$\left(\frac{d^2}{dx^2} + n^2\right) y = 0.$$

$$y = A \cos nx + B.$$

---

$$\frac{d^2y}{dx^2} = ax + by.$$

$$\frac{d^2}{dx^2}(ax + by) - b(ax + by) = 0.$$

$$\therefore \quad ax + by = C e^{\sqrt{b}\,x} + C_1 e^{-\sqrt{b}\,x}$$

---

$$(1 + x^2)\frac{d^2y}{dx^2} + 1 + \left(\frac{dy}{dx}\right)^2 = 0$$

$$\frac{\frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2} + \frac{1}{1 + x^2} = 0.$$

$$\therefore \tan^{-1}\left(\frac{dy}{dx}\right) + \tan^{-1} x = \tan^{-1} C.$$

$$\tan^{-1}\frac{\frac{dy}{dx} + x}{1 - x\frac{dy}{dx}} = \tan^{-1} C.$$

$$\therefore \frac{\frac{dy}{x} + x}{1 - x\frac{dy}{x}} = C.$$

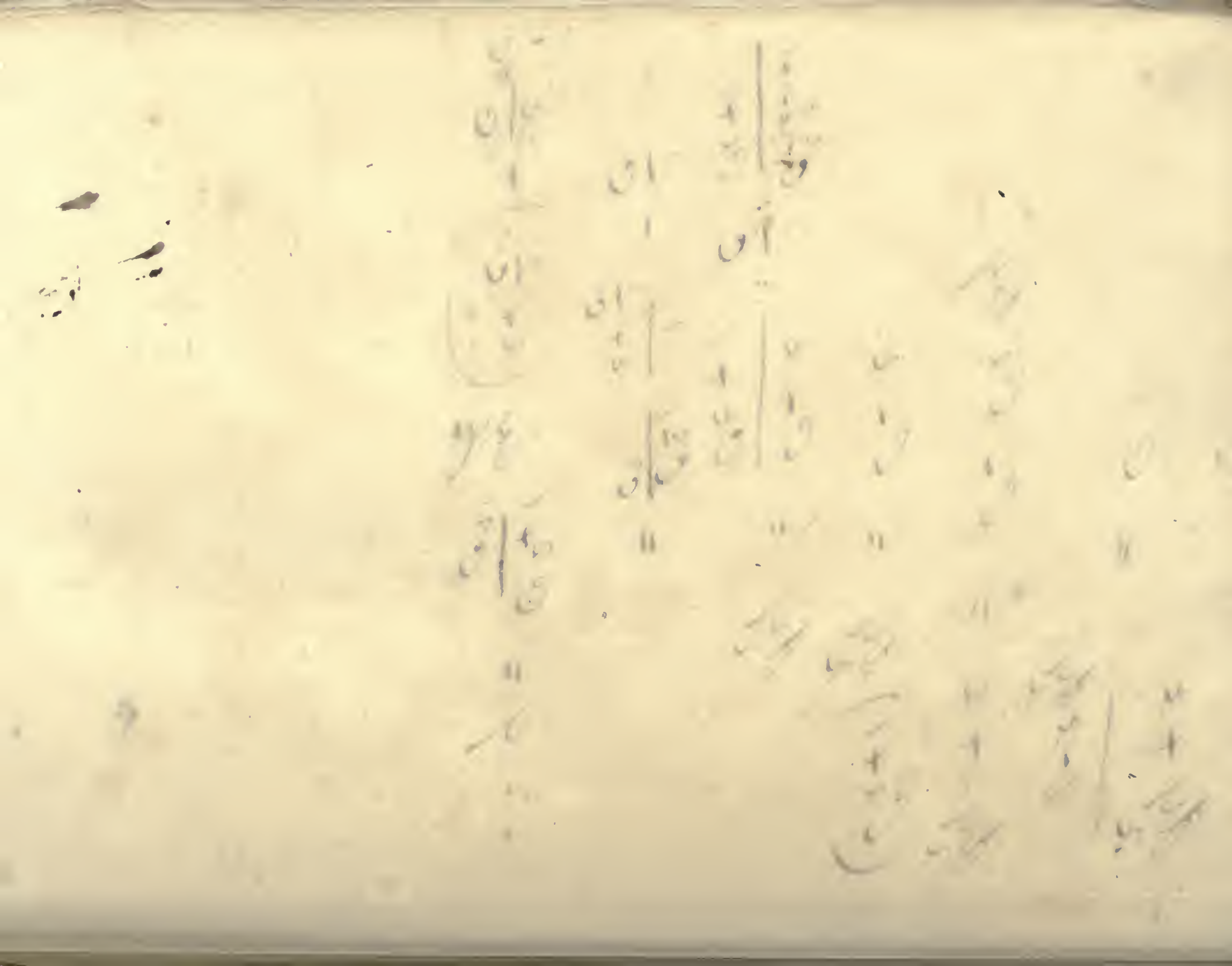
$$\frac{dy}{x} + x = C - Cx\,\frac{dy}{x}.$$

$$(Cx + 1)\frac{dy}{x} = C - x.$$

$$\frac{dy}{x} = \frac{C - x}{Cx + 1} = \frac{1}{C}\cdot\frac{C^2 - Cx + 1}{Cx + 1}$$

$$= \frac{C^2 + 1}{C^2}\,\frac{1}{x + \frac{1}{C}} - \frac{1}{C}$$

$$\therefore y = \frac{C^2 + 1}{C^2}\log\left(x + \frac{1}{C}\right) - \frac{x}{C} + C$$



$$P \cdot d_x^2 y + \frac{1}{2} \frac{d}{x} P \cdot d_x y + C_y = 0 \cdot \quad \{ P = f(x)$$

$$P 2 d_x y \cdot d_x^2 y + \frac{dP}{x} \cdot (d_x y)^2 + 2 C_y d_x y = 0 \cdot$$

$$d_x \{ P \cdot (d_x y)^2 \} + 2 C_y \cdot d_x y = 0$$

$$P \cdot (d_x y)^2 + C_y^2 + C' = 0 \cdot$$

$$\frac{y.dx - x\,dy}{y\sqrt{y^2 - x^2}} = \int \frac{y.dx - x\,dy}{y^2.} \cdot \frac{1}{\sqrt{1 - \frac{x^2}{y^2.}}}$$

$$= \frac{d.\left(\frac{x}{y}\right)}{\sqrt{1 - \frac{x^2}{y^2.}}} = 0.$$

$$\therefore \sin^{-1}\frac{x}{y} = C.$$

$$\frac{x^{n-1}}{y^{n+1}}\left(ny\,dx - nx\,dy\right) = 0.$$

$$d\left(\frac{x^n}{y^n}\right) = 0.$$

$$\therefore \frac{x^n}{y^n} = C.$$

Reduce $(a+bx)^2 \frac{d^2y}{dx^2} + A(a+bx)\frac{dy}{dx} + By = f(x)$

to a linear eqn with constant coeffts.

Let $a+bx = z$.

$\therefore b^2 z^2 \cdot \frac{d^2y}{dz^2} + Abz \cdot \frac{dy}{dz} + By = f(x)$.

Let $z = e^{-t}$.

Then $\frac{dy}{dz} = \frac{dy/dt}{dz/dt} = -\frac{1}{z}\frac{dy}{dt}$.

$$\frac{d^2y}{dz^2} = \frac{d^2y/dt^2}{(dz/dt)^2} - \frac{dy/dt \; d^2z/dt^2}{(dz/dt)^3}$$

$$= \frac{d^2y/dt^2}{z^2} - \frac{dy/dt}{z^2}.$$

$\therefore z^2 \cdot \frac{d^2y}{dz^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$.

$z \cdot \frac{dy}{dz} = -\frac{dy}{dt}$.

$\therefore$ the eqn $= u$ becomes.

$$b^2\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + Ab\frac{dy}{dt} + By = f(x)$$

$$\frac{d^2y}{dt^2} - \left(\frac{A}{b}+1\right)\frac{dy}{dt} + \frac{B}{b^2}y = \frac{1}{b^2}f(x).$$

in wh. the coeffts are constant.

$$x.(a+bx)\,d_x^2 y + (c+ex)\,d_x y = 0.$$

$$\frac{d_x^2 y}{d_x y} + \frac{c+ex}{x(a+bx)} = 0$$

$$\frac{d_x^2 y}{d_x y} + \frac{e}{2b}\left\{\frac{2bx+a}{ax+bx^2}\right\} + \frac{c-\frac{ae}{2b}}{b\left\{\frac{a^2}{4b^2}+\frac{ax}{b}+x^2-\frac{a^2}{4b^2}\right\}}$$

$$\log. d_x y + \frac{e}{2b}\log.(ax+bx^2) + \frac{c-\frac{ae}{2b}}{b}\int_x \frac{1}{\left(x+\frac{a}{2b}\right)^2-\left(\frac{a}{2b}\right)^2}$$

$$\log. d_x y + \frac{e}{2b}\log.(ax+bx^2) + \frac{c-\frac{ae}{2b}}{a}\log.\left(\frac{2bx+2a}{2bx}\right) = \log C.$$

$$\frac{cx^{-2}}{ax^{-1}+b} + \frac{e}{a+bx}$$

$$\frac{d^2y}{dx^2}\frac{d^3y}{dx^3} = a.$$

$$q\frac{dq}{dx} = a.$$

$$q^2 = 2ax + C$$

$$dp_x = \sqrt{2ax + C}.$$

$$p = \frac{dy}{dx} = \frac{1}{3a}\left(2ax + C\right)^{\frac{3}{2}} + C'$$

$$y = \frac{1}{3a}\cdot\frac{1}{5a}\left(2ax + C\right)^{\frac{5}{2}} + C'x + C''$$

$$y^3 \cdot d_x^2 y = c$$

$$d_x^2 y = \frac{c}{y^3}.$$

$$2 dy_x \cdot d_x^2 y = \frac{2c\, dy_x}{y^3}.$$

$$(dy_x)^2 = -\frac{c}{y^2} + C'$$

$$= \frac{C' y^2 - c}{y^2}.$$

$$\frac{dx}{y} = \frac{y}{\sqrt{C' y^2 - c}}.$$

$$\therefore x = \frac{1}{C'} \sqrt{C' y^2 - c} + C''.$$

$$\frac{d^2y}{dx^2} + n^2 y = 0$$

$$y = A \cos nx + B \sin nx$$

when $x = 0$ $y = 1$ $\therefore B = 0$

$$1 = A \cos 0$$

but $\cos 0 = 1$ $\therefore A = 1$

$$y = \cos nx$$

$$\frac{dy}{dx} = -n \sin nx = 0 \text{ if } x = 0$$

$\therefore y = \cos nx$ satisfies the required condition

4. $\dfrac{\cot\frac{\theta}{2^x}}{2^x} = \dfrac{\tan\frac{\theta}{2^{x+1}}}{2^{x+1}}$.

Express $\Delta^n u_x$ in terms of $d_x^n u_x$ and successive differences.

Prove. $\Delta^n u_x = \log.(1+\Delta)^n u_x$.

---

$$\frac{\cot\frac{\theta}{2^x}}{2^x} = \frac{\cot\frac{\theta}{2^{x+1}}}{2^{x+1}} - \frac{\cot\frac{\theta}{2^x}}{2^x} = \frac{2^x}{2^x \cdot 2^{x+1}}\left\{\cot\frac{\theta}{2^{x+1}} - 2\cot\frac{2\theta}{2^{x+1}}\right\}$$

$$= \frac{1}{2^{x+1}}\left\{\cot\frac{\theta}{2^{x+1}} - \left\{\frac{1-\tan^2\frac{\theta}{2^{x+1}}}{\tan\frac{\theta}{2^{x+1}}}\right\}\right\}$$

$$= \frac{1}{2^{x+1}\tan\frac{\theta}{2^{x+1}}}\left\{X - X + \tan^2\frac{\theta}{2^{x+1}}\right\} = \frac{\tan\frac{\theta}{2^{x+1}}}{2^{x+1}}.$$

---

$$\Delta^n u_x = u_x + n.\Delta u_x + \frac{n(n-1)}{1\cdot 2}\Delta^2 u_x + \&c.$$

$$= u_x + nh.\frac{\Delta u_x}{h} + \frac{nh(nh-h)}{1\cdot 2}\frac{\Delta^2 u_x}{h^2} + \&c.$$

$h$ decrease indefinitely while $nh = t$ remains constant.

then $\frac{\Delta u_x}{h}$ the limiting ratio of the diff. of $u_x$ and the increment $= d_x u_x$

$$\frac{\Delta^2 u_x}{h^2} = d_x^2 u_x \quad \&c.$$

$$\therefore \Delta^n u_x = u_x + t d_x u_x + \frac{t^2}{1\cdot 2}.d_x^2 u_x - \&c. + \frac{t^n}{\lfloor n}.d_x^n u_x.$$

$$= \left\{1 + t d_x + \frac{t^2 d_x^2}{1\cdot 2} + \&c.\right\} u_x = \epsilon^{t d_x} u_x.$$

$$\left(x/r - y\right)\left(x/r - 2y\right) + x^3 = 0.$$

$$x^2/r^2 - 3xy/r + 2y^2 + x^3 = 0$$

$\therefore$ for a Lemniscate Polar.

$$2x^2/r - 3xy = 0.$$

$$r = \frac{3y}{2x}.$$

$$\frac{9y^2}{4} - \frac{9}{2}y^2 + 2y^2 + x^3 = 0.$$

$$-\frac{y^2}{4} + x^3 = 0.$$

$$y^2 - 4x^3 = 0$$

$$(y - x \cdot d_x y)^2 = b^2 + a^2 (d_x y)^2$$

$$y^2 - 2xy\, d_x y + x^2 (d_x y)^2 = b^2 + a^2 (d_x y)^2$$

$$V = y^2 - 2xy/p + (x^2 - a^2)/p^2 - b^2 = 0$$

$$\frac{d V}{p} = -2xy + 2\cdot(x^2 - a^2)/p = 0$$

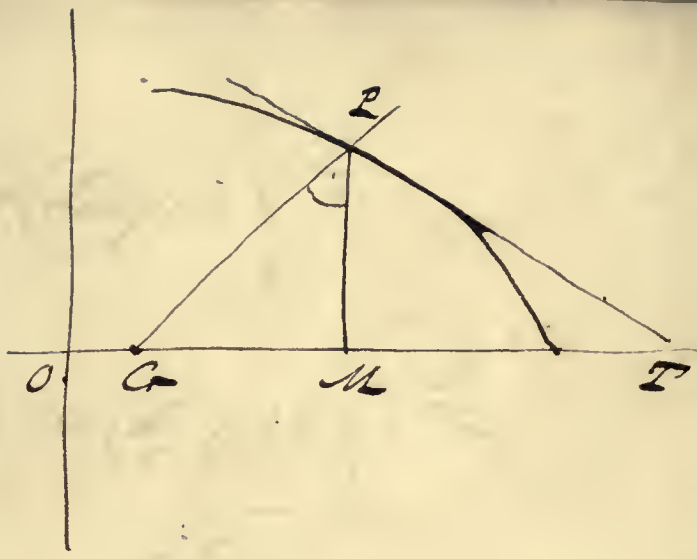
$$\therefore p = \frac{xy}{x^2 - a^2}$$

$$\therefore y^2 - \frac{2x^2y^2}{x^2 - a^2} + \frac{x^2y^2}{(x^2 - a^2)} - b^2 = 0.$$

$$y^2 - \frac{x^2y^2}{x^2 - a^2} - b^2 = 0.$$

$$-a^2y^2 = b^2(x^2 - a^2)$$

$$a^2(b^2 - y^2) = b^2x^2.$$




$\ell$ of the normal =

$$y\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

$$\rho = -\frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$\ell$ of the normal = rad of curv where.

$$y\sqrt{1+\left(\frac{dy}{dx}\right)^2} + \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = 0.$$

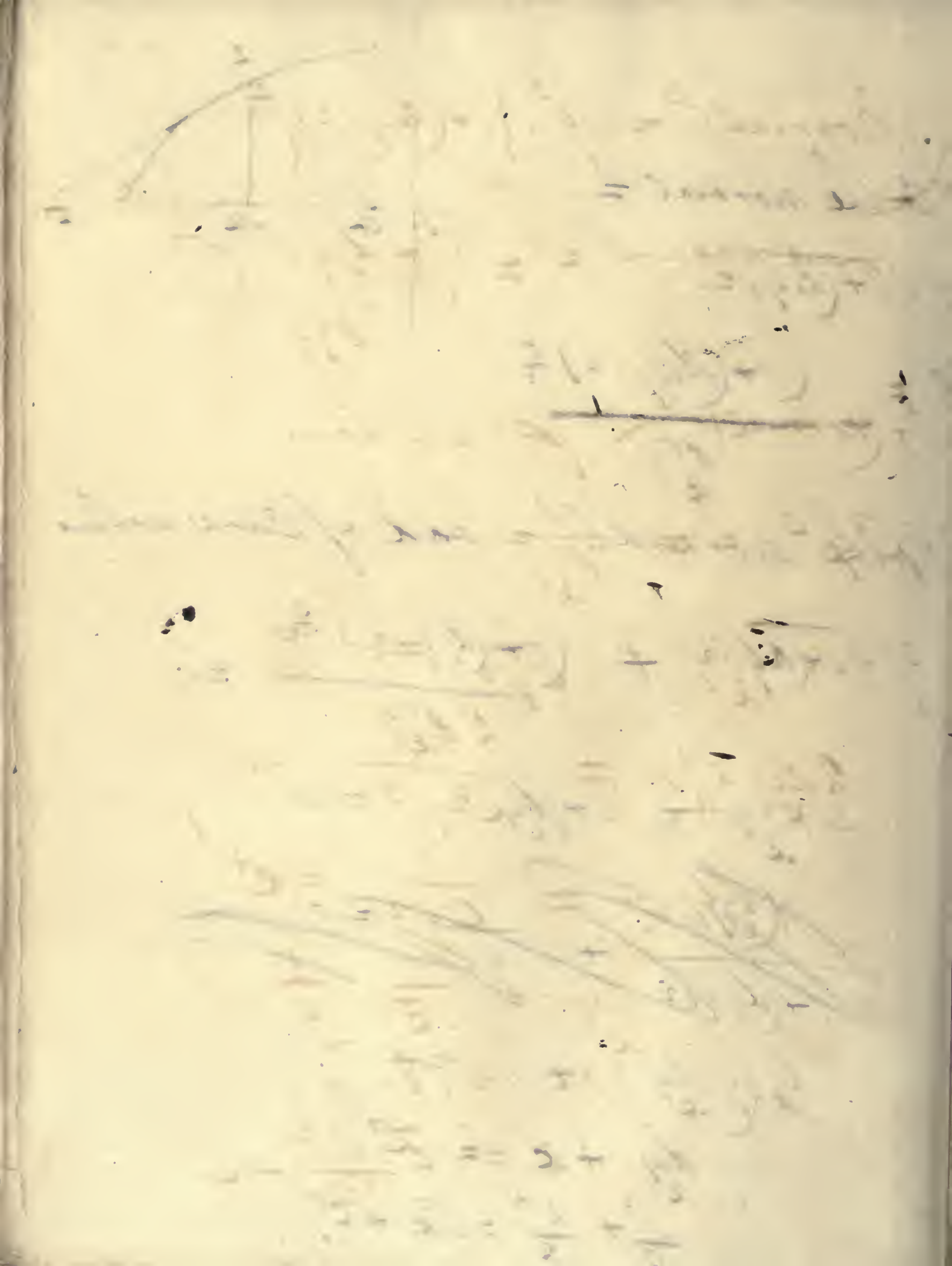
$$y\frac{d^2y}{dx^2} + 1+\left(\frac{dy}{dx}\right)^2 = 0.$$

$$\frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2} + \frac{1}{y} = 0.$$

$$\frac{d}{dx}\left(y\cdot\frac{dy}{dx}\right) + 1 = 0.$$

$$y\cdot\frac{dy}{dx} + x = C.$$

$$\frac{y^2}{2} + \frac{x^2}{2} = Cx + C_1.$$



$$(\text{normal})^3 = y^3 \cdot \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}.$$

$$\text{Rad of curv} = \pm \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

If $(\text{normal})^3 \propto p$ - reciprocal.

$$Cy^3 = \pm \frac{1}{\frac{d^2y}{dx^2}}$$

$$\therefore \frac{d^2y}{dx^2} \pm \frac{1}{Cy^3} = 0.$$

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} \pm \frac{2\frac{dy}{dx}}{Cy^3} = 0$$

$$\left(\frac{dy}{dx}\right)^2 \mp \frac{1}{Cy^2} = C'$$

$$\left(\frac{dy}{dx}\right)^2 = C \pm \frac{1}{Cy^2}$$

$$= \frac{C'Cy^2 \pm 1}{Cy^2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{C}} \cdot \frac{1}{y}$$

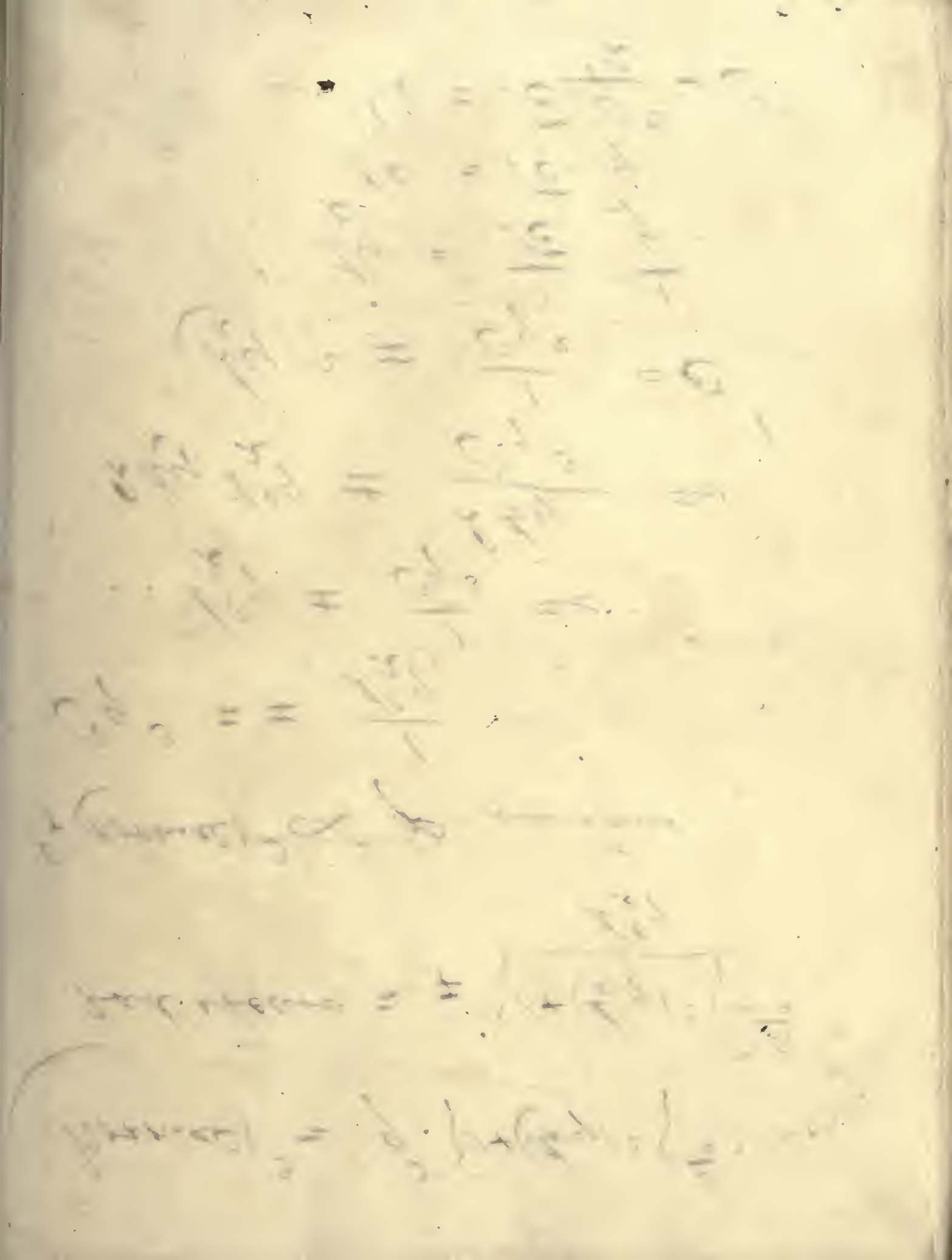
$$= \frac{1}{\sqrt{C'}\sqrt{Cy^2 \pm 1}}$$

$$dx = \sqrt{C} \cdot y \cdot$$

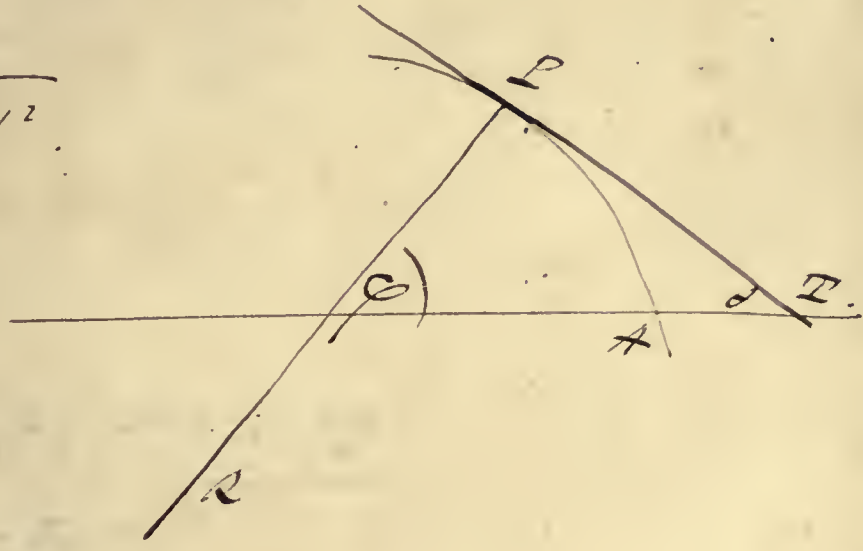
$$x = \frac{1}{\sqrt{C'}}\sqrt{y^2 \pm \frac{1}{CC'}}$$

$$x = \sqrt{C}\frac{y^2}{2} + C'$$

$$x^2 = y^2 \pm \frac{1}{CC'}$$



Find a curve in which radius of curv. $\propto \operatorname{cosec}^3\phi$.




$\operatorname{cosec}\phi = \sec\theta = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$.

$\operatorname{cosec}^3\phi = \left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}$.

$\rho = -\dfrac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$

$\therefore \rho = \mu \operatorname{cosec}^3\phi$

$$\frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \mu\cdot\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}.$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{\mu}.$$

$$\therefore y = \frac{x^2}{2\mu} + C_1x + C_2.$$

$$z = f\left(\frac{x-z}{y-z}\right) = f(v) \text{ suppose.}$$

$$d_x z = d_v \cdot f(v)\, d_x v.$$

$$d_y z = d_v f(v)\, d_y v.$$

$$v = \frac{x-z}{y-z}.$$

$$d_x v = \frac{(y-z)(-d_x z) + (x-z)\, d_x z}{(y-z)^2}$$

$$= \frac{y-z + (x-y)\, d_x z}{(y-z)^2}.$$

$$d_y v = \frac{(x-y)\, d_y z - x-z}{(y-z)^2}.$$

$$\therefore \frac{d_x z}{y-z+(x-y)\, d_x z} = \frac{d_y z}{(x-y)\, d_y z - (x-z)}$$

$$(x-y)\, d_x z\, d_y z - (x-z)\, d_x z =$$

$$(y-z)\, d_y z + (x-y)\, d_x z \cdot d_y z.$$

$$(y-z)\, d_y z + (x-z)\, d_x z = 0.$$

The centre of a ○ moves along a line $y = mx + c$ its rad. increases so as always to be $\mu$ times the abscissa of its centre; find the curve which envelopes the ○s.

Let $x, y$ be the coords of the cen. of the ○

$\therefore$ the eq$^n$ of the ○ is $(X-x)^2 + (Y-y)^2 = c'^2$

but $c' = \mu x$, $\quad y = mx + c$




$\therefore \left(X - \frac{c'}{\mu}\right)^2 + \left(Y - \frac{mc'}{\mu} - c\right)^2 = c'^2. \qquad (A)$

Differentiate with respect to $c'$

$$\frac{1}{\mu}\left(X - \frac{c'}{\mu}\right) + \frac{m}{\mu}\left\{Y - \frac{mc'}{\mu} - c\right\} + c' = 0.$$

$$\left(1 - \frac{1}{\mu^2} - \frac{m^2}{\mu^2}\right)c' + \frac{m}{\mu}(Y - c) + \frac{X}{\mu} = 0.$$

$$c' = \frac{m\mu(Y + X - c)}{m^2 + 1 - \mu^2}.$$

substitute in $A$ and we have.

$$\left(X - \frac{m(Y+X-c)}{m^2+1-\mu^2}\right)^2 + \left(Y - \frac{m^2(Y+X-c)}{m^2+1-\mu^2}\right)^2 = \left(\frac{m\mu \cdot Y+X-c}{m^2+1-\mu^2}\right)$$

$$\{(m^2-\mu^2)X - mY + c\}^2 + \{(1-\mu^2)Y - m^2X + m^2c\}^2 = m^2\mu^2\{Y+X-c\}$$

In a conic section. $\rho = \frac{(\text{normal})^3}{(\text{semi lat. rect})^2}$

Let $y^2 = nx + mx^2$ be the eqn for a conic sec$^n$.

$\therefore \rho = -\frac{1}{d_x^2 y}\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}$.

also the normal $= y.\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}}$.

$\therefore \rho = -\frac{1}{d_x^2 y}\ \frac{(\text{normal})^3}{y^3}$.

$2y\frac{dy}{dx} = n + 2mx$. $\therefore \frac{dy}{dx} = \frac{n+2mx}{2y}$.

$2y\,d_x^2 y + 2\left(\frac{dy}{dx}\right)^2 = 2m$. $\therefore d_x^2 y = \frac{m - \left(\frac{dy}{dx}\right)^2}{y}$.

$d_x^2 y = \frac{4my^2 - \{n^2 + 4mnx + 4m^2x^2\}}{4y^3} = -\frac{n^2}{4y^3} + \frac{4my^2 - 4mnx - 4m^2x^2}{4y^3}$

$= -\frac{n^2}{4y^3}$ for $4m(y^2 - nx - mx^2) = 0$

$\therefore \rho = \frac{4y^3}{n^2}.\frac{(\text{normal})^3}{y^3} = \frac{(\text{normal})^3}{\left(\frac{n}{2}\right)^2}$

and $\frac{n}{2}$ is the semi-latus rectum.

$$u = \int_{x_1}^{x_2} \sqrt{1+p^2}\,dx$$

$$\delta u = \int_{x_1}^{x_2} \left\{ \left( \sqrt{1+p^2} + \frac{p\,\delta p}{\sqrt{1+p^2}} \right)(1 + d_x \delta x) - \sqrt{1+p^2} \right\}$$

$$= \int_{x_1}^{x_2} \frac{p\,\delta p}{\sqrt{1+p^2}} + \sqrt{1+p^2}\, d_x(\delta x)$$

$$= \int_{x_1}^{x_2} p \cdot \frac{(d_x \delta y - p\, d_x \delta x)}{\sqrt{1+p^2}} + \sqrt{1+p^2}\, d_x \delta x$$

$$= \int_{x_1}^{x_2} p \cdot \frac{(d_x \delta y - p\, d_x \delta x + p\, d_x p \cdot \delta x)}{\sqrt{1+p^2}} + \sqrt{1+p^2}\, d_x \delta x$$

$$= \int_{x_1}^{x_2} \left\{ p \cdot \frac{d_x(\delta y - p \delta x)}{\sqrt{1+p^2}} + d_x(\delta x \cdot \sqrt{1+p^2}) \right\}$$

$$= \delta x \cdot \sqrt{1+p^2} + (\delta y - p \delta x) \frac{p}{\sqrt{1+p^2}} - \int_{x_1}^{x_2} (\delta y - p \delta x)\, d_x \left( \frac{p}{\sqrt{1+p^2}} \right)$$

$$= - \int_{x_1}^{x_2} (\delta y - p \delta x)\, d_x \frac{p}{\sqrt{1+p^2}}$$

since at limits $\delta y$ $\delta x$ are equal zero

$$\therefore d_x \left( \frac{p}{\sqrt{1+p^2}} \right) = 0$$

$$\frac{p}{\sqrt{1+p^2}} = c$$

$$\therefore p = c_1$$

$$y = c_1 x + c_2$$

Calculus of variation.
Find the shortest curve joining two points.

$$u = \int_{x_0}^{x} \sqrt{1+p^2}\,dx$$

$$\delta u = \int_{x_0}^{x} \left( \sqrt{1+(d_x y')^2}\cdot d_x x' - \sqrt{1+p^2} \right)$$

$$= \int_{x_0}^{x} \left\{ \sqrt{(d_x x')^2 + (d_x y')^2} - \sqrt{1+p^2} \right\}$$

$$= \int_{x_0}^{x} \left\{ \sqrt{(1+d_x \delta x')^2 + (p + d_x \delta y)^2} - \sqrt{1+p^2} \right\}$$

$$= \int_{x_0}^{x} \left\{ \sqrt{1+p^2} + \frac{d_x \delta x + p \cdot d_x \delta y}{\sqrt{1+p^2}} \right.$$

In the $n^{th}$ a rectangular trajectory of a system of
Curves compared with the $n^{th}$ $r_1 r_2 = c^2$, $r_1 r_2$ being radii vectors
drawn from the pole to any point in the curve, and c being
the variable parameter.

$2a$ = distance between the poles. Take the origin at the
bisection of this distance. Then

$$r_1^2 = r^2 + a^2 + 2ar\cos\theta.$$

$$r_2^2 = r^2 + a^2 - 2ar\cos\theta.$$

$$\therefore r_1^2 r_2^2 = c^4 = (r^2+a^2)^2 - 4a^2r^2\cos^2\theta.$$

$$= r^4 + a^4 - 2a^2r^2\cos 2\theta.$$

$$0 = r^3 - a^2r\cos 2\theta + a^2r^2\sin 2\theta\,\frac{d\theta}{dr}.$$

If this be any point of the curve $\frac{r}{dr} = \varphi(r,\theta)$

$$0 = r^2 - a^2\cos 2\theta + a^2\sin 2\theta \cdot \varphi(r,\theta)$$

but $r,\theta$ being a point in the trajectory curve

$$0 = 1 + \frac{r}{dr}\varphi(r,\theta)$$

$$0 = r^3 - a^2\left(r\cos 2\theta + \sin 2\theta\frac{dr}{d\theta}\right)$$

$$0 = r^3\,d\theta - a^2 dr\sin 2\theta.$$

$$\therefore \frac{a^2 dr\sin 2\theta}{r^3\sin^2 2\theta} = \frac{d\theta}{\sin^3 2\theta} \quad \therefore \frac{a^2}{2r^2\sin 2\theta} = -\int \frac{1}{\sin^2 2\theta} = -\int_\theta \frac{\sin^2 2\theta + \cos^2 2\theta}{\sin^3 2\theta}$$

$$\therefore \frac{a^2}{r^2} = 2\cos 2\theta - \sin^2 2\theta \int_\theta (\tan\theta)^{\frac{3}{4}} + c\sin^2 2\theta.$$

$$\left(\frac{d\theta}{t}\right)^2 = \frac{2fg}{t}\left(\frac{1}{a\cos\alpha} - \frac{1}{b\cos\alpha}\right) - \frac{3}{2}f\left(\frac{1}{a}\right.$$

$$\therefore \frac{\sec\alpha}{\log\frac{\theta}{2}\cos\alpha} = -\frac{2}{2}\log\tan\left(\frac{\theta}{4}\right).$$

$$= \frac{1}{a}\cdot\log\tan\left\{\frac{(\theta-45^\circ)}{2}\right\}.$$

$$\left(\frac{d\theta}{t}\right)^2 = \frac{2fg}{t}\left(\frac{1}{a\cos\theta} - \frac{1}{b\cos\theta}\right) - \frac{3}{8}f\cdot\frac{1}{a}\cdot\log\tan\left(\frac{\theta-45^\circ}{2}\right)$$

$$-\frac{1}{b}\tan\frac{\theta}{2}.$$

$$= \frac{3fg}{a}.$$



$$g\sin\frac{a}{2}\cdot(\cos\alpha - \cos\theta)\cos\theta$$

$$d^2\theta \quad g\sin\frac{a}{2}\cdot(\cos\alpha - \cos\theta)\cdot\theta.$$

$$g\sin\frac{a}{2}(\cos\alpha\cos\theta - \sin\alpha\sin\theta) + 1$$

$$g\sin\frac{a}{2}\cdot\cos(\theta+\alpha) + 1$$

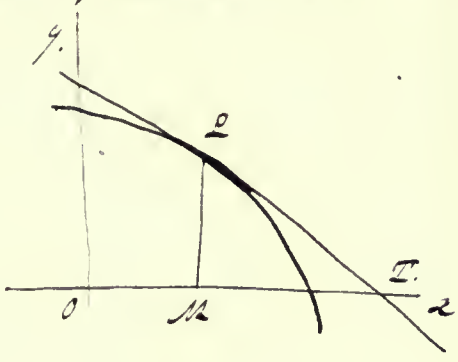
$$v\frac{dv}{ds} = g\sin\frac{a}{2}\cos(\theta-\alpha) + 1$$

$$v^2 = c^2 - 2gx.$$

$$\frac{d^2u}{t} = -g\cdot\cos\alpha - \cos\theta.$$

Diff. = n.

Find a curve in wh. the subtangents a multiple of the abscissa.




$y = f(x)$ the req'd curve.

O the orgn. $OM = x$. $MP = y$. coords of P.

Then. subtangent $MT = -y \cot PTM$.

$$= \frac{-y}{\frac{dy}{dx}}.$$

Absolute abscissa $= OM = x$.

$$\therefore mx = -\frac{y}{\frac{dy}{dx}}.$$

$$\therefore \frac{dy}{y} = -\frac{dx}{mx}.$$

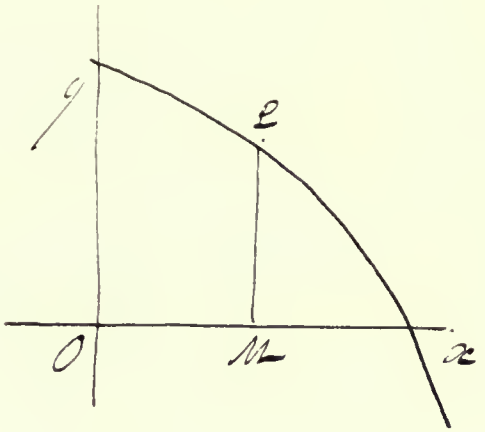
$$\log y = -\frac{1}{m}\log x = \log\left(\frac{1}{x}\right)^{\frac{1}{m}} + \log C.$$

$$\text{or } y = \left(\frac{C}{x}\right)^{\frac{1}{m}}.$$

$$y^m = \frac{C}{x}.$$

Body

Exg. P. 486. Ex. 2.




Let y or M 2 be the given area.

$\therefore A = \int_x y.$

Also the rectangle contd. by the ordinate and the abscissa is $dy$.

$$\therefore \int_x y = m.xy.$$

$$\frac{1}{mx} = \frac{y}{\int_x y}.$$

$$\frac{1}{m}\log x + \log C = \log \int_x y.$$

$$\therefore (Cx)^{\frac{1}{m}} = \int_x y.$$

$$\therefore \left( y = \frac{mC^{\frac{1}{m}}}{m+1} x^{\frac{m+1}{m}} \right)$$

$$y = C' x^{\frac{m+1}{m}}.$$

$$y = C^{\frac{1}{m}} \cdot \frac{1}{m} x^{\frac{1-m}{m}}.$$

$$= C' x^{\frac{1-m}{m}}.$$

Exercises. p. 437. Ex. 3. Ans.

$$p = \frac{y - x\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

$$\therefore x^2\left\{1+\left(\frac{dy}{dx}\right)^2\right\} = y^2 - 2xy\frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^2.$$

$$x^2 = y^2 - 2xy\frac{dy}{dx}.$$

Let $y^2 = vz$.

$$\therefore x^2 = vz - x\cdot d_x(vz).$$

$$= vz - xv\,d_x z - xz\frac{dv}{dx}.$$

$$vz - xv\,d_x z = 0, \quad \frac{d_x z}{z} = \frac{1}{x}.$$

$$\log z = \log x - \log c.$$

$$z = \frac{x}{c}.$$

But $z\frac{dv}{dx} + x = 0.$

$$\frac{x}{c}\frac{dv}{dx} + x = 0.$$

$$\frac{dv}{dx} = -c$$

$$v = -c_1 x + c_2.$$

$$\therefore y^2 = zv = -c_1 x^2 + c' x.$$

$$\text{or } y^2 + x^2 = c'x.$$

pair of 3 white and two black balls, two are placed at random in one bag, and 3 in the other, after which a white ball is accidentally lost: find the probability that a person going to either bag will draw a white ball.

There are 10 combinations of 5 things taken 3 together. The probability of there being 3 white balls in the large bag is $\frac{1}{10}$

2 white and one black. — $\frac{6}{10}$.

1 white and 2 black — $\frac{3}{10}$.

∴ if the white ball had not been lost, the probability of drawing a white ball would have been. $\frac{3+2.6+3}{30} = \frac{3}{5}$

Also there are 10 combinations of 5 things taken 2 together. The probability of there being 2 white balls in the small bag is: $\frac{3}{10}$

1 white and 1 black — $\frac{6}{10}$

2 black — $\frac{1}{10}$.

∴ under the same circumstances as before, the probability of drawing a white ball would be. $\frac{2\times 3+6}{20} = \frac{12}{20} = \frac{3}{5}$. as in the case of the large bag, and as would also have been the case if the balls had not been separated.

∴ The probability of drawing a white ball from either bag after one white one was lost, would be either if the balls all remained together, or if separated into 2 bags. $\frac{2}{4}$ or $\frac{1}{2}$.

1 2 3

4 5

$$\sqrt{a+\sqrt{b}} = \sqrt[4]{\alpha} + \sqrt[4]{\beta}.$$

$$\sqrt{a-\sqrt{b}} = \sqrt[4]{\alpha} - \sqrt[4]{\beta}.$$

$$\sqrt{a^2-b} = \sqrt{\alpha} - \sqrt{\beta}.$$

$$\therefore a^2 - b = \alpha - 2\sqrt{\alpha\beta} + \beta.$$

$$a^2 + 2a\sqrt{b} + b = \alpha + 4a^{\frac{3}{4}}\beta^{\frac{1}{4}} + 6\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} + 4a^{\frac{1}{4}}\beta^{\frac{3}{4}} + \beta.$$

$$\therefore 2(a+\sqrt{b})\sqrt{b} = 4a^{\frac{1}{4}}\beta^{\frac{1}{4}}\left(a^{\frac{1}{2}} + 2\alpha^{\frac{1}{4}}\beta^{\frac{1}{4}} + \beta^{\frac{1}{2}}\right)$$

$$(a+\sqrt{b})\sqrt{b} = 2.\alpha^{\frac{1}{4}}\beta^{\frac{1}{4}}.\left(\sqrt[4]{\alpha} + \sqrt[4]{\beta}\right)^2$$

$$\therefore b = 4\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}.$$

$$a^2 - b = \alpha - 2\sqrt{\alpha\beta} + \beta$$

$$\therefore a^2 = \alpha + 2\sqrt{\alpha\beta} + \beta.$$

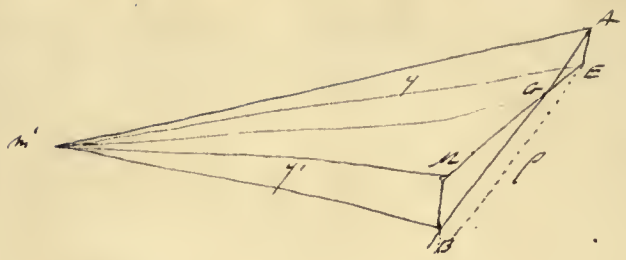
$$\therefore 1 - \frac{a^2}{b} = \frac{b-a^2}{b} = \frac{4\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} - \alpha - 2\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} - \beta}{4\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}}$$

$$= -\left(\frac{\sqrt{\alpha}+\sqrt{\beta}}{2\alpha^{\frac{1}{4}}\beta^{\frac{1}{4}}}\right)^2.$$

$\therefore 1 - \frac{a^2}{b}$ is a perfect square.

If $\sqrt{a \pm \sqrt{b}} = \sqrt[4]{\alpha} \pm \sqrt[4]{\beta}$

Lunar Theory.




$y^2 = (m'A)^2 + AE^2$

$= r'^2 + 2r'GA\cos\overline{\theta-\theta'} + GA^2 + s^2GA^2$

$\therefore \frac{1}{y^3} = \frac{1}{r'^3}\left\{1 - \frac{3GA}{r'}\cos\overline{\theta-\theta'}\right\}$ nearly.

similarly $\frac{1}{y'^3} = \frac{1}{r'^3}\left\{1 + \frac{3GB}{r'}\cos\overline{\theta-\theta'}\right\}$ nearly.

$\therefore \frac{1}{y'^3} - \frac{1}{y^3} = \frac{3}{r'^4}\left\{GA + GB\cos\overline{\theta-\theta'}\right\} = \frac{3\rho}{r'^4}\cos\overline{\theta-\theta'}$.

---

$\frac{1}{\sqrt{1+s^2}}\left\{\frac{MG}{y'^3} + \frac{EG}{y^3}\right\} = \frac{AG}{y'^3} + \frac{EG}{y^3} = \frac{M\rho}{\mu r'^3}\left\{1 - \frac{3M\rho}{\mu r'}\cos\overline{\theta-\theta'}\right\} + \frac{E\rho}{\mu r'^3}\left\{1 + \frac{3E\rho}{\mu r'}\cos\overline{\theta-\theta'}\right\}$

$= \frac{\rho}{r'^3}\left\{1 + \frac{3(E-M)\rho}{\mu r'}\cos\overline{\theta-\theta'}\right\} = \frac{\rho}{r'^3}$ nearly.

$P = \frac{\mu}{\rho^2(1+s^2)^{\frac{3}{2}}} + m'\left\{\frac{MG}{\sqrt{1+s^2}\,y'^3} + \frac{EG}{\sqrt{1+s^2}\,y^3} - r'\cos\overline{\theta-\theta'}\left(\frac{1}{y'^3} - \frac{1}{y^3}\right)\right\}$.

$= \frac{\mu}{\rho^2(1+s^2)^{\frac{3}{2}}} + \frac{m'\rho}{r'^3} - \frac{3m'\rho}{r'^3}\cos^2\overline{\theta-\theta'}$

$= \frac{\mu}{\rho^2}\overline{(1+s^2)^{\frac{3}{2}}} + \frac{m'\rho}{r'^3} - \frac{3m'\rho}{2r'^3}\left\{1 + \cos 2\overline{\theta-\theta'}\right\}$.

$= \frac{\mu}{\rho^2}\left(1 - \frac{3}{2}s^2\right) - \frac{m'\rho}{r'^3}\left\{\frac{1}{2} + \frac{3}{2}\cos 2\overline{\theta-\theta'}\right\}$. nearly.

$S = \frac{\mu s}{\rho^2(1+s^2)^{\frac{3}{2}}} + \frac{m's}{\sqrt{1+s^2}}\left\{\frac{MG}{y'^3} + \frac{EG}{y^3}\right\}$

$= \frac{\mu s}{\rho^2(1+s^2)^{\frac{3}{2}}} + \frac{m's\rho}{r'^3}$ nearly

$\therefore S - Ps = \frac{3m'\rho}{2r'^3}\left\{1 + \cos 2\overline{\theta-\theta'}\right\}$.

$T = -m'r'\sin\overline{\theta-\theta'}\left(\frac{1}{y'^3} - \frac{1}{y^3}\right) = -m'r'\sin\overline{\theta-\theta'}\left\{\frac{3\rho}{r'^4}\cos\overline{\theta-\theta'}\right\}$.

$= -\frac{3m'\rho}{2r'^3}\sin 2\overline{\theta-\theta'}$ nearly.

Let $R = \frac{1}{\sqrt{1-2ph+h^2}}$.

$$\frac{dR}{dp} = \frac{h}{(1-2ph+h^2)^{\frac{3}{2}}}$$

$$\therefore (1-p^2)\frac{dR}{dp} = \frac{h(1-p^2)}{(1-2ph+h^2)^{\frac{3}{2}}}$$

$$\frac{d}{dp}\left\{(1-p^2)\frac{dR}{dp}\right\} = -\frac{(1-2ph+h^2)2ph + 3h^2(1-p^2)h}{(1-2ph+h^2)^{\frac{5}{2}}}$$

$$= h\left\{\frac{3h - 3hp^2 - 2p + 4p^2h - 2ph^2}{(1-2ph+h^2)^{\frac{5}{2}}}\right\}$$

$$= h\left\{\frac{3h + p^2h - 2p - 2ph^2}{(1-2ph+h^2)^{\frac{5}{2}}}\right\}$$

Also $\frac{dR}{dh} = \frac{p-h}{(1-2ph+h^2)^{\frac{3}{2}}}$.

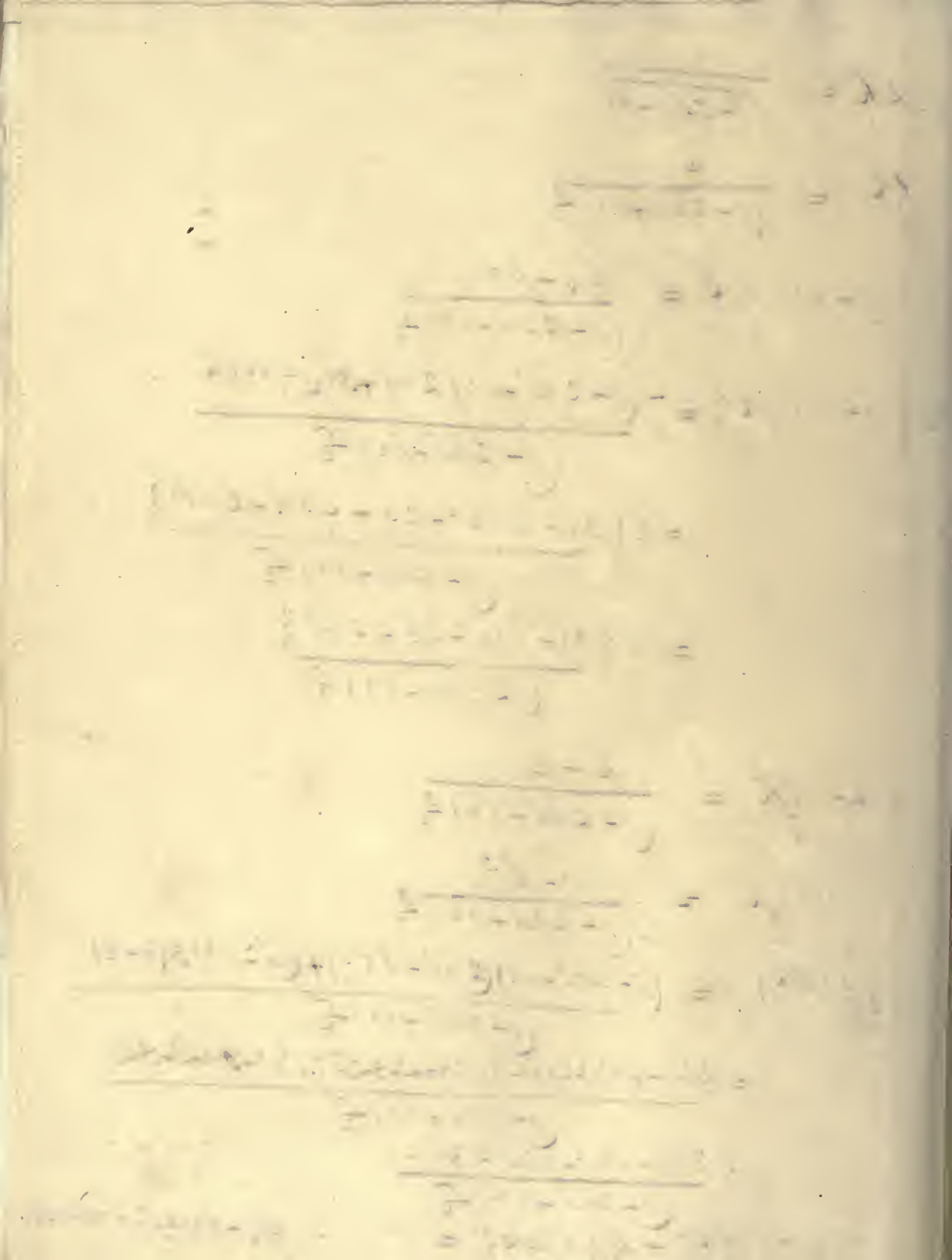
$$h^2\frac{dR}{dh} = \frac{ph^2 - h^3}{(1-2ph+h^2)^{\frac{3}{2}}}$$

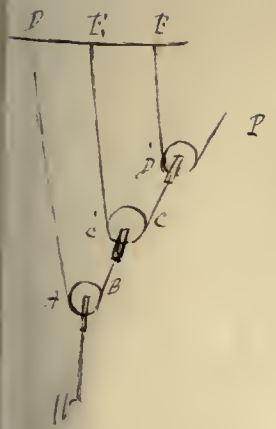
$$\frac{d}{dh}\left(h^2\frac{dR}{dh}\right) = \frac{(1-2ph+h^2)(2ph-3h^2) + (ph^2-h^3)3(p-h)}{(1-2ph+h^2)^{\frac{5}{2}}}$$

$$= \frac{2ph - 4p^2h^2 + 2ph^3 - 3h^2 + 6ph^3 - 3h^4 + 3p^2h^2 - 6ph^3 + 3h^4}{(1-2ph+h^2)^{\frac{5}{2}}}$$

$$= \frac{2ph - p^2h^2 + 2ph^3 - 3h^2}{(1-2ph+h^2)^{\frac{5}{2}}}$$

$\therefore \frac{d}{dp}\left\{(1-p^2)\frac{dR}{dp}\right\} + \frac{d}{dh}\left\{h^2\frac{dR}{dh}\right\} = 0$. $\therefore \frac{d}{dp}(1-p^2)\frac{dP_n}{dp} + n(n+1)P_n = 0$.






If the strings passing round the pulleys A, B, C be inclined to the vertical at angles $\theta, \theta', \theta''$ &c.

The pressure at $B = \frac{W}{2\cos\theta}$

$$C = \frac{\text{pressure at } B}{2\cos\theta'} = \frac{W}{2^2\cos\theta\cos\theta'}$$

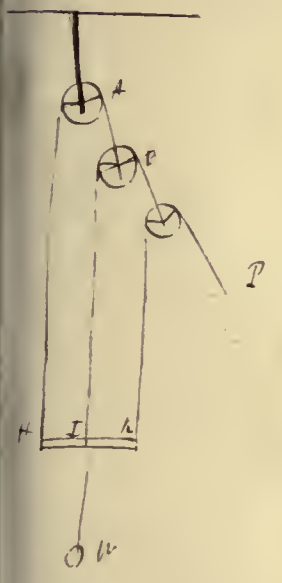
$$P = \frac{\text{pressure at } C}{2\cos\theta''} = \frac{W}{2^3\cos\theta\cos\theta'\cos\theta''}$$

And in the same manner if $n$ be the number of moveable pulleys

$$P = \frac{W}{2^n \cos\theta\cos\theta'\cos\theta''\ \&c\ \cos\theta^{(n)}}$$

$$P = \frac{W}{2^n \cos^n\theta} \Bigg\} \text{ if } \theta = \theta' = \theta'' \text{ \&c}$$

---




$P$ = pressure at $P$. = pressure at $K$

$P\cos\theta$ = pressure at $B$. = ——— $I$

$2^2 P\cos\theta\cos\theta'$ = pressure at $A$ = ——— $H$

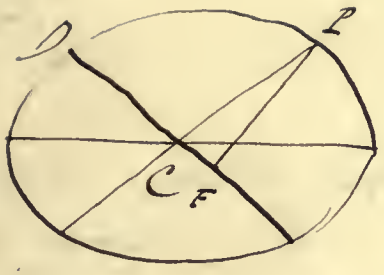
Similarly if $n$ be the number of moveable pulleys the pressure on the last string =

$2^n P \cos\theta\cos\theta' \&c\ \cos\theta^{(n)}$

when $\theta = \theta' = \theta''$ &c ———

$$W = 2P\left\{\cos\theta + 2\cos^2\theta + 2^2\cos^3\theta + \&c\ 2^{n-1}\cos^n\theta\right\}$$

$$\therefore P = \frac{W}{\left\{2\cos\theta + 2^2\cos^2\theta + 2^3\cos^3\theta \&c\ 2^n\cos^n\theta\right\}}$$




Let CP . CD be = semi conjugate diameters.

Cht of Curv. at P. Thyph $= \frac{2CD^2}{CP}$

Semi of Curv at P $\frac{2CD^2}{PF}$

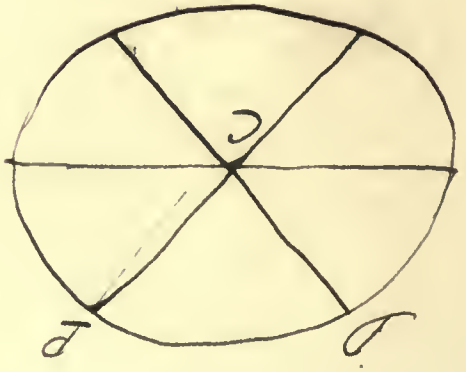
$\therefore$ Cht of Curv × Semi of Curv $= \frac{4CD^4}{CP.PF}$

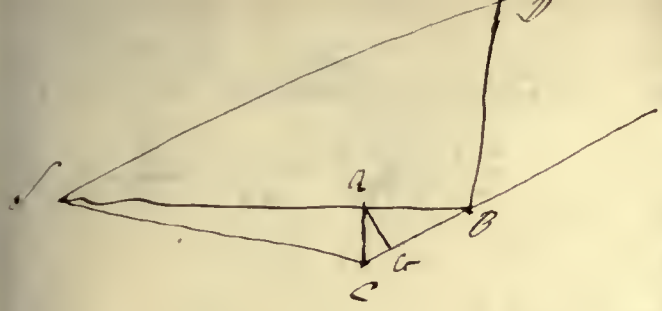
$= \frac{CD^4 + 2CD^2.CP^2 + PC^4}{CD.PF}$ if $CD = PC$

$= \frac{(CD^2 + CP^2)^{1/2}}{CD.PF}$

$= \frac{(a^2+b^2)^{1/2}}{ab}$




C
P
P




As CB latter direct. of mot. before the change of mot.
BD the direct after mot. $BD = 2CB - SB = a$

Then area of $\triangle SCB = \frac{SB.BC}{2} \sin 2\theta = \frac{SB.BC}{4}$.

area of $\triangle SBD = \frac{1}{2} SB.BD = \frac{1}{2} SB.BC$.

$\therefore$ the

$CD : RD$.

$CB + CA : RD :: RB : CB - CA$

$CB^2 - CA^2 = RB^2$

---

Let the body unmoved by S move in the straight line $CB$. In $SB$ this is the direct after and $CR$ the velocity in the direct $RB$ and $CR$, then are $\therefore$ the sides of the $\triangle CBR$.

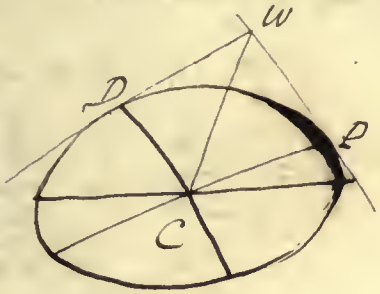
then $BR^2 = CB^2 - RC^2 = (CB + RC)(CB - RC)$

$\therefore (CB + RC) : BQ :: BQ : (CB - RC)$

and this being true will be true in the limit when the force and acts on the body causing it to move in the curve

$\therefore$ the velocity in the direct of the next vector is a mean $\therefore$ to the latter the direct diff of the

If $C, C'$ are the curvatures at pairs of $\odot$ whose tangents are $\perp$ $C^{\frac{2}{3}} + C'^{\frac{2}{3}}$ is constant.




Let the tangents at D and P meet at right angles in W.
C the cent. of the $\odot$ CD and CP the semi conjugate diams at D and P are always $\perp$ to the tangents at their extremity $\therefore$ CP is $\perp$ to CD.

Let $C$ = Curvat at P. $C'$ Curvat at D

$$C = \frac{2CD^2}{CP^3}, \qquad C' = \frac{2CP^2}{CD^3}.$$

$$\therefore C^{\frac{2}{3}} + C'^{\frac{2}{3}} = 2^{\frac{2}{3}}\left\{\frac{CD^{\frac{4}{3}}}{CP^{\frac{2}{3}}} + \frac{CP^{\frac{4}{3}}}{CD^{\frac{2}{3}}}\right\}$$

$$= 2^{\frac{2}{3}}\left\{\frac{CD^2 + CP^2}{CP.CD^{\frac{2}{3}}}\right\}$$

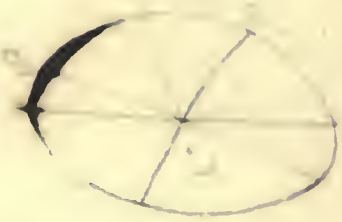
$$= 2^{\frac{2}{3}}\ \frac{a^2+b^2}{(ab)^{\frac{2}{3}}}$$ which is constant.

$C \propto \frac{PF}{CD^3} \propto \frac{Ar.\,\beta c}{-D^3}$

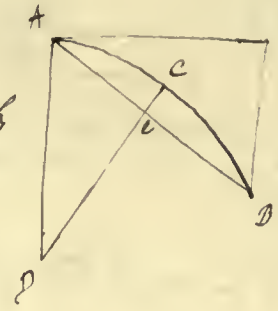
$C^{\frac{2}{3}} \propto \frac{1}{CD^2}$

$C'^{\frac{2}{3}} \propto \frac{1}{CP^2}$ $\therefore C^{\frac{2}{3}} + C'^{\frac{2}{3}} \propto \frac{1}{CD^2} + \frac{1}{CP^2}$

$$\propto \frac{CD^2 + CP^2}{CD^2.CP^2}$$



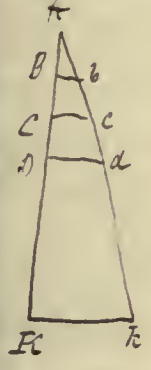
$\angle$ ACB be part of a circle. Centre S. Intersect
ACD in C in DC. this will evidently intersect AB.
wherefore since AB ultimately = ACB, AC ultimately
= AC. Let $\angle ADC = \theta$.




Then $\frac{\sin\theta}{\theta} = \frac{\frac{Ac}{AD}}{\frac{AC}{AD}} = \frac{Ac}{AC} = 1$

when $\theta$ equals 0.

1. At the beginning of the eleventh page when it is stated
area ABD : area Abd :: $AD^2 : Ad^2$ it is necessary that the ordinates
should be // to each other : Leretter Construction is introduced that
all the ordinates should be drawn making any other given angle with AE
Xcept unless the angle about which the curve and the straight line meet were
such the angle at the vertex of the $\Delta$ is here compared would not
be finite




Let Ak be a straight line. AB . BC &c = intervals. Bb . Cc &c
the velocities acquired at the end of the first, second &c intervals
Then by similar $\Delta$s AB : Bb :: AC : Cc.

F = $\frac{\text{velocity acquired in any time}}{\text{time}}$ $\therefore v = ft$

if $\propto$ diff $t$'s constant

$F \propto \frac{Bb}{AB}$ =

$F = \frac{Bb}{AB} = \frac{Cc}{AC}$ =

$F \propto \frac{\text{space}}{\text{time}^2}$ . $\frac{\text{space described in time } AB}{\text{space in } AC} = \frac{AB \cdot Bb}{AC \cdot Cc}$

But Bb : Cc :: AB : AC by similar $\Delta$.

$= \frac{AB^2}{AC^2}$

$F \propto \frac{\text{space}}{\text{time}^2} = \frac{AB^2}{AC^2} \cdot \frac{AC^2}{AB^2} = 1$ which is constant.

Mechanical problem. St John's May. 1836.

Having given the ratios of two moving forces acting upon two unequal masses, also the ratio of the spaces described, and of the velocities acquired find the proportion of the masses.

A bent lever has arms whose lengths are — the proportion of m, and whose densities are — in the like ratio, find the ratio of m. n when the lever rests with its leaves inclined 60° to the horizon. the angle between the arms is 135°.

A weight is the force upon an isosceles Δ is to be supported by means of two unequal rods of given length and weight.

A uniform ladder rests with one end on a rough horizontal plane, and the other against a smooth vertical wall, making a given angle of elevation. Suppose friction to take just sufficient to maintain the equilibrium — what position find what additional force will be requisite to restore the equilibrium — a man of given weight being supposed to ascend.

There are two balls whose elasticity is e. A impinges obliquely on B with a given velocity. find the angles made by A and B after impact with A's original direction of motion when A's deviation is the greatest possible. tan $\theta = \frac{1}{2}\sqrt{\frac{1+e}{1-e}}\sqrt{\frac{1-e}{2}}$ θ being A's deviation. It appears from the investigation that the deviation is independent of the original velocity. In practice however this is not the case, for ordinary billiard balls for instance. account for this difference.

A ladder whose base is an isosceles Δ is laid with its axis in the intersection of two given inclined planes, at right angles to each other. the points of contact with the planes being the extremities of the unequal side. find the position of equilibrium and show that if the equilibrium be stable $\sin^2 \theta < \frac{1}{2}$, a being the base of the triangle and b the other side.

A horizontal shaft is both uniformly and alternately raised and depressed through a given space by a wheel revolving uniformly acting underneath required the form of the wheel. — an ellipse. the point being placed in one of the foci.

Required the size of the wheel which a horse of given size acting at a given distance from the axis will just draw most easily over a given obstacle, the weight of the wheel being given.

A body P is drawn upwards by another weight Q acting over a fixed pulley in a direction which is given to P. required the number of feet described by P during the ascent.

Two bodies of unequal mass are projected in vacuo with velocities v, v' at different angles of elevation in different vertical planes. show that the path of the centre of gravity is a parabola and find its latus rectum.

Find the centre of gravity of a segment of an ellipsoid contained between two parallel planes being given to two principal axes.

Mechanics. Mich. Term. 1835.

1. If a pendulum be slightly altered in length, find the number of oscillations gained or lost in a day. Ex. the Seconds pendulum length 39.13 inches when lengthened .01 inches.
2. AB is a uniform heavy beam moveable about A in a vertical plane. D is a fixed pulley vertically over A. W a weight fastened to a string which passes over D. and is fastened to B. find the position of equilibrium.
3. Find the conditions of equilibrium of any number of forces acting in any direction upon a rigid body. State the modifications which they undergo, when the body is moveable (1) round a fixed axis, (2) round an axis along which can slide.
4. A body acted on by any forces, rests on the plane of xy upon its base, the curve boundary of which is $y = fx$. find the conditions of equilibrium.
5. A body consisting of two straight arms AC. BC making an angle of 150° with other balances itself on a rough horizontal plane. when AC lies upon the plane a weight W is placed at the extremity of CA. and the arm CB has attached a fine point by a weight P which has fallen through 5 feet finds how high W will ascend.
6. If a force f acts in the plane xy at a point x.y. upon a rigid body moveable round the axis of z and inclined at an angle $\alpha$ to the axis of x. shew that $f(x \sin\alpha - y \cos\alpha)$ represents the moment of f to turn the body round the axis of z.
7. Find the c.g. of the catenary between horizontal and vertical co-ords.
8. If a flexible thread be acted upon by forces which are not in the same plane, and if $\phi$ be the resultant of all the forces acting at a point P making an angle $\theta$ with the tangent, and an angle $\psi$ with the radius of absolute curvature and if $t$ be the tension of the string at that point shew that

$$\frac{dt}{ds} = f\cos\theta, \quad \text{and} \quad t = \int f \cos\psi \, ds$$

9. Investigate the formula for finding the centre of gravity of any curve figure, and apply it to find that of a hemisphere.
10. Find the eq.s of motion of a body moving in a plane and acted on by any forces in that plane. Hence shew that a body describing the arc of any curve and from it is acted on at every point by a force which varies inversely as the radius of curvature.

A body acted on by a central force, find the polar eqn. to the curve
scribed viz. $\frac{d^2u}{d\theta^2} + u - \frac{P}{h^2u^2} = 0$ and shew that the velocity at any given distance
m dependent of the curve described.

When a body descends along the arc ($s$) of a curve by the action of gravity
show that the pressure at the point $R = g\frac{dy}{ds} + \frac{1}{\rho}\left(\frac{ds}{dt}\right)^2$, $\rho$ being the radius of
curvature at the point. Ex. When a body oscillates in a cycloid find
that the tension of the string = the weight of the body.

State the principle of virtual velocities — apply it to determine the position
when weight P sustained upon a sphere by another weight Q with which it
is connected by a string passing over a pulley in the vertical diameter
of the sphere produced.

Shew that the moment of inertia of a plane body round an axis through its
centre of gravity is less than round any // axis. Find the moment of
inertia of a body in the form of an ellipse about an axis through its
centre & perpendicular to its plane.

Parallel forces act at the middle points of the sides of a triangle
prove that the resultant passes through the centre of gravity of the
triangle.

Two masses acting bodies connected by a string which passes
over a fixed pulley C, if P is raised until PC is horizontal & is then released
find the motion of P.

Diff Cal. S. Johns. June 1835.

before a multiple point, and shew how to find the angles which the different branches of a curve make with the axis of $x$ at a multiple point. Ex. $y^6 - ay^2x^2 + x^6 = 0$

Investigate Lagrange's theorem for expressing $f(y)$ in a series of powers of $x$ where $y = z + x.\phi(y)$. Ex $(1+xy^2)^n$.

Integrate $\frac{1-x^2}{(1+x)^2(x^2+a^2)}$ ; $\frac{1}{(a+b\cos x)^2}$ ; $(\cos mx . \cos nx)^2$

Integrate $\frac{1}{\sqrt{(a^2-y^2)(b^2-x^2)}}$ by an elliptic function, and prove that $\int_0^\infty e^{-x^2} = \frac{1}{2}\sqrt{\pi}$

Shew that the area bounded by the curve whose eq. is $xy = \frac{a^2}{2}\log_e \frac{a^2}{a^2-x^2}$ and by its asymptote, and by the axis of $x$ is $\frac{\pi^2 a^2}{12}$.

Prove that there are $n$ arbitrary constants in the complete integral of a diff. eq. of the $n^{th}$ order, and $n$ arbitrary functions in the complete integral of a partial diff. equation of the $n^{th}$ order.

Integrate the eqns.

$$D_x y + Py = Qy^n \; ; \quad y = x\int f D_x y + \phi D_x y \; ; \quad P D_x z + R D_y z = R.$$

Prove that $\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-c^2\sin^2 x}} = \frac{\pi}{2}(1+c_1)(1+c_2)\ldots(1+c_n)$.

where $c_1, c_2 \ldots$ are derived from $c$ and from one another by the law $c_{r+1} = (1-\sqrt{1-c_r^2}) \div (1+\sqrt{1-c_r^2})$, and the process is continued till $c_n = 0$ nearly.

Let $D_x^n y + p_1 D_x^{n-1} y + p_2 D_x^{n-2} y + \ldots p_n y = X$ be a linear eq. of the $n^{th}$ order with constant coeffts. and let $-a_1, -a_2, \ldots -a_n$ be the roots of its auxiliary eq. so that

$$m^n + p_1 m^{n-1} + p_2 m^{n-2} + \ldots p_n = (m+a_1)(m+a_2)\ldots(m+a_n)$$

then if we multiply both sides of the proposed eq. by $e^{ax}$, and integrate, the result will be a linear eq. of the $(n-1)^{th}$ order with the same auxiliary eq. except that it wants the factor $(m+a_n)$

Apply the above theorem to shew that the complete solution of the eq. is $y = e^{-a_1 x}\int e^{(a_1-a_2)x}\int e^{(a_2-a_3)x} \ldots \int e^{(a_{n-1}-a_n)x}\int e^{a_n x} X$,

and integrate the equation $-D_x^4 y - y = \cos x$.

Find the complete solution of the eq. $D_x^2 z - v^2 D_y^2 z = 2v^2\left\{\frac{D_y z}{y} - \frac{z}{y^2}\right\}$ by assuming $z = \phi\left(\frac{x}{y}\right)$

12. In the wheel and axle suppose the axle to be rough, and that a perfectly flexible rope is wound round it n times. To the end of which rope is attached a given weight, which is allowed to descend freely, till the whole of the rope left round the axle. To find the velocity acquired and other circumstances when a equilibrium will be established, in order that the weight may descend uniformly after this time, the inertia of the machine being neglected, and the rope being supposed without weight.

13. A uniform flexible rope is suspended from two points in the same horizontal line, find the form in which it will hang, when given weights are suspended at the points of the rope adjacent to the extremity.

14. State and prove D'Alembert's principle, and apply it to find the motion of any other accelerating force on any body round a fixed horizontal axis when acted upon by gravity.

15. To find the proportion of P & W in the first system of pulleys, when the strings are no. //

Defn force Mean. weight reaction. Impress.
Component Resultt.

Defn the resultt. of two forces acting on a particle.

If forces act on a particle in direct & contrary
for the magnitude of resultt.

If the forces act on a pt. x. y. z. in direct // to the
co-ordt axes be $\Sigma(X)$ $\Sigma(Y)$ $\Sigma(Z)$. then that the
= a to the direct of the resultt. $\frac{x'-x}{\Sigma x} = \frac{y'-y}{\Sigma(Y)} = \frac{z'-z}{\Sigma(Z)}$

If a point rel on a plane curve. under forces $X$. $Y$ in the
plane of the curve // to co-ordt axes then that condt.
of = ⁿ $0 = X + Y\frac{dy}{dx}$

for the corresponding condtn. for a particle rel on
a curve — in space. under forces $X$. $Y$. $Z$. if the coordt axes.

the axis of $x$ is vertical, a given particle
rels upon it by a th. passing over a pulley at the
rel. of support & carrying given weight. for the
equilibr. that there may be = ⁿ of about the cond.
of the particle with at the end of the latus rectum

per unpressed force, and state how this measured
Law holds of positive electricity moving in opposite
direct. in the same straight line. Simply for
the subsequent motion

If the two bodies impinge, the motion after
centre of gravity is the same before & after
impact

Two balls lie in contact on a table.
When how one after may be struck; that
the other may move off in a given direction.
A ball drops vertically on two other equal
balls lying in contact on a horizontal plane
& strikes both at once — a vertical
plane through their centres — determine the
subsequent motion.

two balls on a table are connected by a
fine string — if one is projected in a
given manner, find the motion after
string becomes tight.

Given the measure of force when a yard
is unit of space, find its measure when a foot is
the unit

State and prove Galileo's properties.

1. A cable assumes the shape of a parabola when the load is uniformly distributed over the horizontal projection of the cable.

2. The load is uniformly distributed over the surface of a cable when it assumes the shape of a catenary.

If a ball rolls along a straight line, the tangent passes through the point of contact.

The cable assumes any form when the load is uniformly distributed over the horizontal projection of the cable.

Let $w$ be the load per unit length of the cable, $H$ the horizontal tension at the lowest point, $\alpha$ the angle which the tangent at the point of suspension makes with the vertical.

Then $\tan\alpha = \frac{w}{H}\cdot\frac{c}{2}$, where $c$ is the span.

The curve assumed by a cable when the force acts to a cable assumes such a form that the tension at any point varies as the length of the cable from the lowest point.

A uniform flexible string hanging vertically supporting a given load assumes the form of a parabola.

A uniform heavy string hanging freely between two points in equilibrium assumes the form of a catenary.

Let $T$ be the tension at the lowest point, $w$ the weight of unit length of the string, $c$ the span.

Then the tension at any point $= T + wy$, where $y$ is the height of the point above the lowest point.

Let $s$ be the length of the string measured from the lowest point to any point $P(x, y)$.

Then $s = \frac{1}{2}\left(ae^{\frac{x}{a}} - ae^{-\frac{x}{a}}\right)$, where $a = \frac{T}{w}$.

Let $\psi$ be the angle which the tangent at $P$ makes with the horizontal.

Then $\tan\psi = \frac{dy}{dx} = \frac{1}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)$.

Let $\rho$ be the radius of curvature at $P$.

Then $\rho = \frac{ds}{d\psi} = \frac{1}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$.

Hence $\rho = y$.

Let $p$ be the perpendicular from the lowest point on the tangent at $P$.

Then $p = \frac{dx}{ds}\cdot y = \frac{y}{\frac{ds}{dx}} = \frac{y}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = \frac{y}{\sqrt{1+\sinh^2\frac{x}{a}}} = \frac{y}{\cosh\frac{x}{a}}$.

Hence $p = a$.

Let $\theta$ be the angle between the tangent at $P$ and the tangent at the lowest point.

Then $\theta = \frac{s}{\rho} = \frac{s}{y}$.

$R$ = radius of curvature at $P$

$r$ = radius of curvature at $O$

$c$ = distance of centre of curvature from centre of curvature at $O$

If a point be acted on at the same instant by 3 pressures each of wh. varies as the sine of the angle contd. between the lines in wh. the other two act it will rest.

Given a pressure. Reqd. to resolve it into 2 pressures at right angles to each other.

I. a weight W is supported placed upon an inclined plane by a power P which acts over a pulley.




Find the pressure on the plane, and the relation between P and W.

To find the pressure on the fulcrum of a lever kept at rest by any two pressures.

If 3 pressures act in a plane, and keep a body at rest, then I. If each of one of the pressures = the other two II. If their directions meet the line of act. of every pressure pass through some one point.

State the principle of virtual velocities and apply it to shew the condition of equilibrium of a system of smooth tubes in a vertical plane the highest and lowest points the parts of the system being in contact with fixed smooth surfaces.

Find the point of = m upon a given cycloid and verify the result

of a rod of given length. The ends of which are on the given cycloid in the

The lower end of a rod is supported on a curve. The rod rests also on a given smooth peg. Find the nature of the curve that any point may be one of = m.

Define stable and unstable = m. when a body whose surface is spherical rests on upon a sphere. Find the condition of stable = m.

Find the centre of gravity of a circular arc of 60°, of a parabola cut off by a double ordinate of a right cone.

3 methods. 1 the = m of = m

2. the use of virtual velocities

3 other cases in wh. the = m of a body is reducible to the = m of a pt.

Before the moment of a couple can be known the axis is a proper definition of the couple
Show that the sum of a couple may be treated about any point - its own plane unchanged allowing its effecting state. Prove the ∠ of couples.
Forces 1-3-5 of 7 act at equal distances // about a rigid rod - find the single = of couple.
Find the magnitude and direction of a system of forces acting in one plane on a rigid body.
Show in what case the forces will not be reducible to a single resultant.
The centre of a cone rests on a horizontal table.
Find the moment of a couple which will keep it with its slant side vertical.
A rectangular ∠ plate is placed on an inclined plane with its length horizontal: if the inclination of the plane is increased below - when the plate will roll and on what point

Forces act on a rigid body in space. For it cannot
therefore may admit a single resultant.

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Forces (2-2, 3-3) act at the opposite corners of a rectangle
for the couple to vanish they are = vl.
∴ a the cond. of = on of any rigid body acted on by ∥ forces.
= m ——— when the sys. is a force
pl. or the body is capable of sliding along the axis of z forces.

When a body capable of turning about an axis
is = m. the forces on the axis is the same as if the forces
were directly applied at it in ∥ directions.

Forces act on a rectangular ⊏ lapel. represented
in magnitude & direct. by 3 edges which neither meet
nor are ∥. for the = of forces to couple. Being whether
the forces admit of a single resultant or not.

Two uniform beams of one in length and weight each
each other in ∠ Inclined planes, and upon 2 vertical planes.
find the pressures on the planes

Before
Two forces acting on a plane on a rigid body, if their
resultant (1) when their directions meet (2) when the
directions are // and opposite. In the latter
case then unless the proof will fail if the forces
are equal.
Def. To resolve a force with regard to an axis
and a plane. Def. the coord. of a system of
// forces. to find the coord. of the resultant.
a ⊥ for its axis vertical. Ex. Given four forces
the vertex a particle with each the coeff. of
friction being given
a particle rests on a plane. attached to the
vertex by a string. How does the tension vary
with the height of the thing, also the pressure
on the plane vary
a particle rests on a rough inclined plane.
under the act. of given forces. Find the
locus of the limiting direction of reaction

$\frac{a}{\sin}$.

$2a \cdot \sin$

$\frac{\sin}{n}$.

Show that any system of forces can always be reduced to 2 single forces - and in the case when it can be reduced to one find the Eq^n of the line in wh: it acts

A body acted on by given forces has 2 points fixed show how to find the forces on these points

Ex: A rect: $ABCD$ is supported by hinges at $A$ & $B$ $AB$ being vertical find the pres: on the hinges.

A convex surface rests on a hor: plane show how to determine whether the $=^m$ is stable or unstable.

Ex: A $\mathfrak{P}$loid the length of whose axis $=^s$ its Lat: rect.

State & prove fundam: pr: & state the 2 purposes to wh: it may be applied. Ex: (1) Find C. of Gr: of the quad: of $\mathfrak{P}$ola & (2) find the vol: of a right cone (3) & surface of a right cone (4) Find $=^n$ to catenary in terms of $x$ & $y$ & show that diff: bet: the Tensions at any 2 points & the vertical dist: bet: them.

Find cond^ns of $=^m$ of a cord (1) on rough $\mathfrak{P}$ola (2) any rough curve & show that if 2 given $=$ Ws sustain ea: other by a string passing over a smooth curve the plane of wh: is vertical the sum of the pres: on any arc depends only on the ord^es of its extremities

Prove that $v = d_s s$ & assuming that $f = d_s v$ show that $v d_s v$ & $\int v = \pm d_s s$ $\int^2 = \pm d_s v$ $\int^3 = \pm d_s^2 s$ $\int = \pm v d_s v$ state when $\pm^n$ & $-^n$ is to be used

Prove the $=^m$ of motion on a $\mathfrak{P}$t referred to 3 rect: axes

Supposing a hole bored to the center of the earth the rad: of earth $= a$ & a body be let fall from a height $h$ above the surface find time of reaching the center & the vel: then acquired. assuming force of Gr: outside the earth to $\propto$ $\frac{1}{\text{dist}^2}$ & inside directly as the distance

$\frac{2\pi}{n}$

(1) Under the action of a central force the hyperbolic power $= \mu/y$ the area described $= \frac{1}{2}\sqrt{\mu a(e^2-1)}\,t$.

(2) Find the $\perp$ to the path orbit described in a $\frac{1}{2}$ of (2)

(3) Force $= \frac{\mu}{\text{dist}^2}$, a body is projected towards the centre of a pt. A such that $SA = a(1+e)$ with a vel: $= \sqrt{\frac{\mu}{a}\frac{(1-e)}{1+e}}$ time of reaching a pt. at a distance $a(1-e)$ from S & vel: Rem: Find also how far the body will go on the other side of A in the time of reaching the latus rectum point.

(4) The force is attractive $\propto \frac{1}{\text{dist}^2}$: A body is projected from A with a vel: $k\sqrt{\mu}$ of in a dir$^n$ $\perp$ to SA apply the $=^{\text{ns}}$ motion to find the vel: at any point & the $=^{\text{n}}$ to the orbit SA = h.

(5) Suppose a body to move uniformly with a vel: c in the curve $y^2 = 4ax$ find $\frac{d^2x}{dt^2}$ & $\frac{d^2y}{dt^2}$ & show that the whole force at any pt. is in the dir$^n$ of the normal at that point.

$$t^{-\frac{1}{2}}dt^{\frac{1}{2}} = t^{\frac{1}{2}} + \frac{1}{2}\int x\,x^{-\frac{3}{2}}dt^2$$

$$a + da \qquad a - da$$
$$SA - SP = 2da$$

$$dt^A = d\theta^A\, dt^\theta$$
$$dt^r = d\theta^r$$

$$r \qquad \frac{1}{r^2}$$
$$r^{-2}d\theta^2 = dt^2 \qquad \frac{h^2}{r^4}$$
$$: dt^2 =$$

If three forces keep a point at rest, any one of them
proportional to the sine of the angle between the direc-
ns of the two others.

A stone dropped from a bridge, strikes the water in
what is the height of the bridge? Also, if the stone
projected downwards, with a velocity of 3 feet per second,
what time will it strike the water?

Two weights sustain each other upon two inclined planes,
ing a common altitude, by means of a string which is
llel to the planes; find their position, taking into
unt the weight of the string which is supposed to be of
form thickness.

If a body, whose elasticity is (e), impinge on a plane
a velocity (v), and be reflected with a velocity (v'); and
, a' be the angles which the directions of its motion, in
two cases, make with a perpendicular to the plane, prove
tan a = e tan a'; v sin a = v' sin a'.

The locus of the centres of gravity of the areas of all
t-angled triangles on the same hypothenuse (2a), is a
e whose radius = $\frac{a}{3}$, The locus of the centres of gravity
their perimeters is a spiral, whose equation is
sin θ − sin $\frac{\pi}{4}$); the pole being in the middle point
the hypothenuse, and θ measured from that line.

hemispherical bowl is terminated by a cylindrical
, having the same external and internal radius;
t must be its breadth, in order that the vessel may
upon any point of its spherical base?

7. A perfectly elastic ball is projected from the middle point of the base of a vertical square, towards one of the angles, and after having been reflected at the sides containing that angle, falls at the opposite angle; find the velocity of projection.

8. Two weights P and Q, connected by a string of given length, are placed upon an inclined plane of the same length, P being at the highest point, and Q at the lowest; after what time will their distance be a minimum?

9. A uniform rod whose length equals twice the diameter, passes through a hole in a spherical shell, and rests with one end against the internal surface; shew that if $a$ be its inclination to the vertical, when in its position of stable equilibrium, $\frac{1}{3}(\pi+a)$ and $\frac{1}{3}(\pi-a)$ will be its inclinations, in its positions of unstable equilibrium.

10. A uniform rod of given length ($a$) is bent into the form of a cycloid, and oscillates about a horizontal line joining its extremities; prove that the length of the isochronous pendulum $= \frac{a}{5}$.

11. A convoy moving uniformly along a road which runs east and west, is perceived at the instant it is due south of a battery; at what elevation, and towards what point of the compass, must a cannon, loaded with a given charge, be fired at the same instant, so as just to hit it?

Reduce the coeffs of refract. from the greatest and Calth. difference

The style of
~~Describe~~ Horizontal dial which is accurately graduated for a given place is called a Dial of 5. A is the Lat. of the place is no reason alluded in the time of the Dial when that is the Lat. of the Lat. of

Angle = $\delta \sin \overline{L-d} \cdot \{ \text{Cot } SL + \text{Cosh. Lat } ZL \}$

Gregory or Cass. Red.

focal length of mirr. 20. 1.

foc length of eyeglass ½ of mirr.

Magnifying power — 800.

---

focal length of mirrors 30. 2 inch

foc length of eyeglass 1 inch.

Magnif 300

---

with Ramsden eyeplace double the magnifying power, with same data incorrectly.

$$m = \frac{2}{3} \cdot \frac{f_o^2}{f_e \cdot \frac{f_e}{m}}$$

Explain briefly of the Astronomical telescope in its
completion consider light homogeneous.
Examine the effect on the magnifying power. the field
of view, on the brightness of a part of the object glass
or eye glass be stopped up.

Explain the compensation which makes Huygens
eye piece achromatic.

Show why Huygens eye piece cannot be
used in a telescope intended for measuring
Describe the Astronomical telescope with
Ramsdens eye piece & find the distance of
the field glass from the focus of the object glass.

In Cassegrains telescope is used with Huygens
eye piece show that the magnifying power
equals $\frac{2}{3} \cdot \frac{f_o^2}{f_e \cdot f_m}$ $f_e$ = focal length of eye glass.

In Gregorian telescope his Ramsdens eye piece
the focal length of large mirror is 9 inches. the
lenses of the eye piece are $\frac{1}{3}$ of an inch apart
and magnifying power is 60 times. find the focal $\frac{169}{450}$ inch
length of the small mirror.

The eye glass of an Astronomical has a focal
length of 2 inch - find in what direction
from it must be moved for any eye
which can see a dist. of 8 inches and
Compare the magnifying power to such an
eye with that of an ordinary case

Describe Microscope & Leading Latent

Hence if a right line drawn from any point in the curve to the centre of force, and a right line drawn from the same point perpendicular to the tangent, be respectively proportional to the velocity and the central force, the curve will be described.

Let the curve be described by the motion of a point P, and let the centre of force be S. Draw SY perpendicular to the tangent PZ, and let the velocity be as PY. Then, because the force is as the velocity and the curvature conjointly, it is as PY and the curvature conjointly. Let the curve be described by the motion of a point P, and let the centre of force be S. Draw SY perpendicular to the tangent PZ, and let the velocity be as PY. Then, because the force is as the velocity and the curvature conjointly, it is as PY and the curvature conjointly.

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2)$$

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$\frac{dv}{dt}$ and from Lemma 7. we have then that

$\frac{dv^2}{2} = \int u \frac{dx}{r} = 0$

State and prove Lemma 9. and then in what part of the proof each limitation of the statement is taken into account.

State and prove Lemma 10. If the curve AK be a right line then that the force is constant. If it be a curve then the radius vector drawn by AK. find the velocity the space described from whatever time.

Then has the curvature of two curves which have a common tangent at the point where they meet any be compared.

If the two curves $y = a^{\frac{2}{3}} x^{\frac{1}{3}}$ and $y = Sbx$. then which has the greater curvature at the origin

Let the circle of curvature which any part of the curve, in from that to the

$$= \lim \frac{(ac \cdot)^2}{\text{subtense} + \text{to tangent}}$$

Hence for the rad of curv at the vertex of a C. also at the vertex of a curve

$$y = \sqrt{\frac{a^2 x + a^3}{b}}$$

the new angular velocity is increased, and is about the cent. of force is now $\frac{1}{dist^2}$

a body is projected and acted upon by a force varying as the dist. the space due to velocity is $\frac{1}{2}$ dist. the angle of projection is 60°. find the orbit and the position.

a body is projected and acted upon by a force varying as $dist^{-2}$. the space due to velocity is $\frac{1}{2}$ dist. the angle of projection 30°. find the axis of the orbit and the point.

a body is describing an ∅ about the focus. what force keeps the absolute force it is obliged to $u+v$. find the consequent change in the major axis and eccentricity.

force $\propto dist^{-2}$. a body falls in a st. line to the cent. of force. find the time of fall, velocity acquired. deduce the velocity acquired from it, and space described.

Two bodies one falls directly to the cent. of force. the other moves in an orbit about it. find the velocities when they are equidistant and equal. shew the equal chords are = described.

when a body describes an ∅ the angular velocity about the focal dist. is double the areal.

a body describes a circle about an internal pt. find the angular velocity. the mean angular velocity.

Condit. of = m of a solid floating freely in a fluid.
The press of air under a given temp. acts inwardly
as the wall it occupies.
pinh the = m of a solid is slightly disturbed from
vertical thro the cen. of grav of the fluid displaced.
Defic the meta cent of a floating body. and show towards
which it tends the stability of equilibr.
To find the met. of a triangular prism floating with an
horizontal and one edge immersed.
To find the met. of a cone floating with its axis vertical
To investigate the relat. betw the density pressure
temp of air

$$\Pi = \rho_T \{ 1 + \alpha t \}.$$

+ $\alpha$ is the diff. bet. 2 states by means of thermom.
in both cases.

Ind the force with wh. a sheet impels a plane. The
~~thing~~ sheet impinges (1) directly (2) obliquely & strikes.
I. d ——————— & the generatg line of a cylinder
impels it (1) in that direct. (2) in direct. through
angle subtended.
——————— called effectual in direct. of fric.
wh impels it in that direct.

A sphere moves thro' a fluid for retarding force
arising from the resistance of the fluid.
Describe the construct. & use of Sykes Hydrometer.

——— Hydraulic Ram.

I is the point of the rudder, on the effect
of the rudder in turning the ship is as sin.
I from a given const. cut off a portion by
a plane // to the base, with the whole resist.
of the ship push. & moving in direct. of sin
shall be the sum. — sin the middle pt of
the sum of the fourth must be equidist. from the
vertex & from the base of the base
Inventions in Paris of the Society of
Encouragement.
Compare the life years of two fluids by weighing
the same vessel — each.
Less. the construct. & use of the atmos. Steam

Hydrostatics

1 Explain fully how the pressure at any point in a mass of fluid is measured; and find the pressure at any point in a mass of fluid at rest acted upon by gravity

2 Find the pressure of a heavy incompressible fluid upon any surface.

3 Describe the experiments by which it appears that at a given temperature the density of air is proportional to its pressure.

4 Compare the specific gravities of two fluids by means of a common hydrometer.

5 Describe Watt's double acting steam engine and the high pressure steam engine.

6 Describe the forcing pumps, and shew how it is employed in a fire engine.

7 Compare the specific gravities of air and water, and state in round numbers the value of the ratio for some given temperature and pressure.

8 A cylindrical vessel containing fluid is inclined till the surface of the fluid passes through the highest point of its base; find the magnitude and point of application of the resultant of the pressure upon the base.

9 State the law which connects the density, temperature, and pressure of atmospheric air; and if a heavy piston confine in a cylindrical vessel a given quantity of air at different temperatures, compare its distances from the base in the two cases.

10 Shew that the principle of virtual velocities obtains in the equilibrium of forces applied to pistons in the sides of a vessel filled with an incompressible fluid.

11 Express the equations of the motion of a fluid in terms of rectangular co-ordinates.

12 Explain what is meant by the metacentre of a floating body, and its position determines the stability or instability of the equilibrium. Find the metacentre of a right-angled cone floating in a fluid of twice its specific gravity

13 Find the pressure at any point in a mass of fluid at rest under the action of any forces; and apply the result to determine the pressure at any point of a fluid contained in an open vessel which revolves uniformly about a vertical axis.

14 Describe the Hydraulic Ram.

1 Define the following terms: "specific gravity", "density", "volume", and "mass". Give the principal equations which connect them.

2 (1) Shew how to find the specific gravity of a body by weighing it in a fluid heavier than itself.

(2) The specific gravities of oil and sea water are respectively 913 and 1030 and that of air 1.2, find the specific gravity of a cone which rests with one-third of its axis in each of the fluids, the vertex being downwards

3 (1) There is a vessel, the external and internal figures of which are similar cones on the same axis; the space between the cones is filled with materials whose density varies as the $n^{th}$ power of their distance from the vertex, measuring along the axis. At what depth will it float in a fluid whose density varies as the $m^{th}$ power of the depth?

(2) The nature of the fluid being the same as in the last question, find the centre of pressure of a rhombus which hangs vertically; having its lesser angle at the surface

4 A hollow semi-paraboloid of given size and weight, is suspended by 3 strings in such a position, that the plane of section coincides with the surface of the fluid. Find the tensions of the strings

5 The transverse section of the bank of a canal, is a portion of a circle, included between the arc, abscissa, and ordinate. Find the conditions that it may not be overturned.

6 (1) Define the Metacentre, and shew how it assists us in discovering the stability or unstability of a floating body. If the specific gravity of a cone be to that of a fluid :: 1 : 2, shew that in order that the equilibrium may be stable $\alpha > 2\sec^{-1}\sqrt{\sqrt{2}-1}$, where $\alpha$ is the vertical angle of the cone.

(2) In the above problem, the cone is supposed to float with its axis vertical. Suppose that not to be the case, find the position of equilibrium.

7 Describe accurately the high-pressure steam engine, distinguishing between the high and low pressure.

8 The tube of a common barometer being supposed to be of equal bore throughout, suppose the air at the sealed end to have been imperfectly exhausted, and that under a given pressure with temperature 50°, the difference of the columns of mercury at the two ends was $h_1$; a given quantity of mercury being poured into the open end, the difference of altitude was observed to be $h_{11}$, and afterwards $h_{111}$; $h_{11}$ being the altitude for a moment, in consequence of the air in the tube being heated to 50° + t by compression and $h_{11}$ the altitude, when the air has cooled down to the temperature of the surrounding atmosphere. Find the quantity of air in the tube, and the atmospheric pressure.

When a fluid acted on by gravity is at rest, the pressure is the same at all points in the same
ry ontal plane. whatever be the form of the vessel containing it.
ve the sum of the perpendicular pressures of a fluid on any surface, and then in what case it will be the
e as their resultant. Ex. A cone with its base in the surface of a fluid
what the resultant of the pressure of a fluid on the surface of a solid immersed in it is equal to the weight of the
e displaced. Hence shew that if a semiellipsoidal boat of given weight $W$ floats upon a fluid with $\frac{1}{3}$ of its axis below the
ace a weight ($W$) put into it will make it float with $\frac{2}{3}$ of its axis below the surface.
w how to measure the pressure of the atmosphere; and correct for the temp of the mercury.
te accurately how the boiling and freezing points of a thermometer are determined in which the temperature
Reaumer, which is denoted by the same number according to Fahrenheit and the Centigrade scale.
the common pump, find the tension of the rod the height through which the water rises at each
ke, and the least value of the range of the piston.
scribe Smeaton's air pump, and the barometer gauge.
d the specific gravity of a solid whose weight is less than the weight of the fluid displaced
it by using it in air as a fluid. Ex.
epare the specific gravities of two fluids by means of Nicholson's hydrometer, and
w how the instrument may be employed to determine the weight of a small solid.
scribe the hydraulic ram.
d the difference of altitude of two stations by means of the barometer, taking into
count the variation of gravity in the same vertical
d the metacentre of a cube of wood floating in water with one face horizontal
en specific gravity of wood = $\frac{3}{2}$ specific gravity of water.
d the pressure at any point of a mass of fluid acted upon by given forces.
determine the form of the surface of a heavy fluid in a cylindrical vessel, revolving
iformly about its axis which is vertical, and if there be a solid cylinder floating on the
id and having its axis coincident with that of the vessel find how much it will be depressed.
d the relation between the pressure and velocity in fluid motion when the velocity
any point is independent of the time; and thence deduce the velocity with which
n incompressible fluid acted on by gravity, issues through a very small
ifice in the vessel containing it.
vestigate the four equations of the motion of a fluid in terms of rectangular
o-ordinates, and express them in terms of $\varphi$.

Hydrostatics. St John's. May. 1842.

Then how to find the value of $\int_n$ of a surface
made by a plane thro' the origin
The locus of the Pole of such planes thro' the given
axis 2b - at make with it the plane $\frac{x}{a} = \frac{z}{c}\sqrt{\frac{a^2-b^2}{b^2-c^2}}$ is a ⊙. viz $x^2+y^2=b^2$.
Tangent plane = $\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=1$ & normal at
(x'y'z') at the surface $u = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$.
The number of normals from a point on the $n^{th}$ order that
may fall on the same point connected
$n^3-n^2+n$.
Investigate the formula $\int_x \int_y V = \int_x \int_y S = \int \sqrt{1+p^2+q^2}$
& explain how the limits of the integrat. are fixed.
Find vol of Pod.
Prove $\int_x^a \sin^{-1}\frac{\sqrt{a^2-x^2}}{\sqrt{b^2-x^2}} = \frac{\pi}{2}\left(b-\sqrt{b^2-a^2}\right)$ find area
of spherical surface intercepted by a right cone
whose vertex is centre of a sphere.
Find the = of the common helix. & its
length.
$\int_0^\pi \log(1+2m\cos x+m^2) = 0$ when $m<1$
$= 2\pi \log m$ $m>1$.
$\int_0^\infty \frac{\sin x \cos mx}{x} = \frac{\pi}{2}$ when $m$ lies between $1$ & $-1$
and zero in all other cases.
$\int_0^\infty \frac{x \sin cx}{1+x^2} = \frac{\pi}{2}e^{-c}$ Prove $\int_0^\infty \frac{x}{1+x^2}\tan^{-1}\left(\frac{m\sin cx}{1-m\cos cx}\right)$
$= \frac{\pi}{2}\log\frac{1}{1-me^{-c}}$

define a diametral surface. Show that the surfaces of the 2nd order each is a plane and find its surface.

Every surface of the 2nd order has at least one diametral plane. I take —

Every variety of surface of the 2nd order is comprehended in the $= n$ $Ax^2 + By^2 + Cz^2 + 2Ayz + 2Bzx + 2Cxy + 2Dx + 2Ey + 2Fz + D = 0$.

Investigate the condition of the surface represented by the preceding $= n$ being an ellipsoid or $+$ —

Prove that $x^2$ represents a continuous surface generated by revolution about the axis $x = y = z$.

and Cayley's real axis $= \sqrt{2}$

$x^2 + 2y^2 + 3z^2 + 4xy + 4xz + 6yz + 2x + 4y + 2z = 1$

represents a continuous and the coordinates of whose centre are $h = k = l$

Prove $1 + \frac{n}{1} a \cos \alpha + \frac{n(n-1)}{1 \cdot 2} a^2 \cos 2\alpha = r^n \cos n\theta$.

$r^2 = 1 + 2a \cos \alpha + a^2$. $\tan \theta = \frac{a \sin \alpha}{1 + a \cos \alpha}$

$\int_0^{\infty} \frac{r^n \cos n\theta}{1 + x^2} = \frac{\pi}{2} \left(1 + a \cdot 2^{-\frac{1}{2}}\right)^n$

before velocity, and state how it is measured.
Explain and prove the first law of motion.
When two velocities are compounded whether they be represented by 2 straight lines forming sides of a ☐ the resulting velocity will be represented by one of the diagonals of the ☐.
If a particle have velocities v. v. in directions of the axes of xyz, find the eqs. to the direct of its motion from the origin.
Also find its velocity in the direction of a straight line whose eq. is $\frac{x}{a} + \frac{y}{b} = 1$
State and prove the 3rd law of motion.
A body urged from rest by a constant force describes 3 feet in 2 seconds. If on a velocity which it will describe 6 feet in 3 seconds, find f.
A force represented in magnitude & direction by the diagonal of a ☐ acts on a particle in succession along the sides of the ☐ taken in order. Find the resultant when the forces represented by the sides are taken in order.
The velocity generated by a gun in a bullet of 1 oz in avoir. lb is 600 ft per second. The bullet is then discharged in a direction making an angle $\frac{\pi}{6}$ with the horizon. Find the moving force supposed uniform that acted on the bullet.

Given the velocity and direction of a projectile, to find its horizontal place & the projectile's vertical place after the projectile's given time of flight.

Given the velocity of projection & the angle of projection, to find the time of flight, and the horizontal range.

Given the time of flight, and the horizontal range, to find the velocity of projection, and the angle of projection.

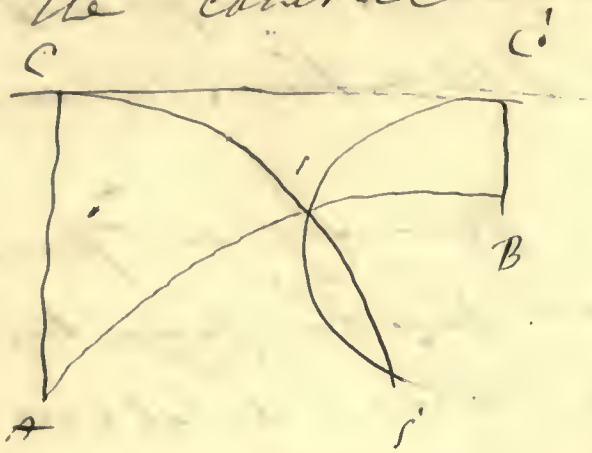
Given the time of flight, and the vertical height, to find the velocity of projection, and the angle of projection.

A ball is projected in a vertical plane & strikes perp. a vertical wall. If the distance from the wall when it again strikes the horizontal plane thro' the point of projection.

Two balls are projected simultaneously from the same point with velocities $v$ & $v'$ at inclinations $\alpha$ & $\alpha'$, then find the time from the instant that the point comes to the ground.

$$= \frac{2}{g}\{v\sin\alpha - (\alpha - \alpha') + v\cos\alpha + v'\cos\alpha'$$

the construction is the same as




Let A be the point of proj. B the point

through which it passes.

Then as the velocity in proj. A C the

direction from direction is known.

with the velocity A C describe

circle, & draw B = rad = BC' from

S S are the points required.

2 parts: velty ℄ in each lever on the same side of the Fulcrum vel: when they are inversely as their distances from the Fulcrum

If 2 wts vel: in one part or a lever they will balance in any part - this time. also when the lever is heavy

A press: P acts at the pt A in a direction making an ∠ α with Ax shew that the resolved pts of P in Ax & Ay ⊥r to Ax are P cos α & P sin α (1st when α lies bet: 0 & 90° (2d) when bet: 180° & 270°

Prove the truth of the principle viz: that when a body is kept in eq: by pressures in one plane the sum of the Moments of the pressures one way round any pt whatever in the plane = the sum of the moments of the press: the other way round the same pt in the case of question (1) the given pt being ½ way bet: the pts of applicatn of P & Weight vel: of P to W on the inclined plane P acting ∥l to the plane & shew that in this case $\frac{P : V.V}{W : V.V} \overset{=}{\leq} \frac{W}{P}$

Weight vel: of P to W when they are eq: in the combination of the Wheel & axle with the single movable pulley. & how that in this case $\frac{P : V.V}{W : V.V} = \frac{W}{P}$

Define the centre of gravity & find that of a pyramid

Define the terms Acc: force moving force & Vel: & give their measures assuming the Acc: force of gr: to be 32.2 when a second is the unit of time & the foot when a minute is the unit

State the 3d law of motion & the ∥gm of Vels & point out the ~~Parts~~ diff: bet: the 2d law of motion & the ∥gm of Vels & the purposes to wh: they are respectively applied

Shew that the motion of the centre of gravity of 2 bodies moving in 1 plane is not affected by Impact

A body is projected with a vel: of 161 ft: in a direction making an ∠ cos: $\frac{4}{5}$ with the horizon find how high it will rise & how long. find also its perp: distn & vel: after 3" & also after 6".

When a pressure P produces motion in a body whose wt is W find the Acc: force & the space it will drag it thro in t"

Shew that the space due to the Vel: at any pt of the path of a projectile is the distance of that pt from the directrix

(141 Find the greatest height of the projectile above a given horizontal plane through the point of projection

A body is projected upwards in a direction with a velocity $u$, how after describing a space $s$ has a velocity $v$. Shew that $v^2 = u^2 \pm 2gs$.

For $\sqrt{2gs}$ is the velocity which gravity would generate from rest through a space $s$, and it is true that $v = u \pm \sqrt{2gs}$

whereas the eq: $v = u \pm ft$ is true

The height of a room is 10 feet. a body of elasticity $\frac{1}{2}$ is projected upwards, and after striking the ceiling and floor just rises to the ceiling again. Find the velocity of projection.

Two pieces bodies connected by a string move over a fixed pulley. find the space which either describes in $t$" from rest.

Either of which connect. Can be more and. in their position after $t$" more.

A body slides down an inclined plane. Shew that the velocity which it acquires is the same as if it fell down freely the same height.

Divide an inclined plane into n parts, so that the time of descent down each after any of the same.

B is a body lying on a table, and P string, passing over a pulley, to the string becomes which determine the subsequent motion of P and Q

to the hyperbola parallel to $HL - SL = 2a$.

Then has a point of the curve may be described & a conic to the polar eqn. to the hyperbola from focus the same trace the curve

find the eqn. to the hyperbola and its position with the point of contact is shown in the diagram.

Use it to prove the fact that it bisects the external angle between the focal distances.

Draw a tangent line to hyperbola from a given external point.

Find the $\mathcal{K}$ conjugate to a given line $\parallel$ to it & find its eqn. referred to the axes

Of a pair of conjugate diam. only one can meet the hyperbola

$$\frac{SL}{H} \quad CZ^2 - CD^2$$

$$SL \cdot HL = CD^2$$

All $\square$ whose sides are $\parallel$ to $\mathcal{K}$ and two conjugate of the infinity of pairs of conjugate diam. are equal in area and Hamilton diagonal coincident with asymptotes.

The locus of the vertex of a $\triangle$ on a given base and having the base & the difference of the sides always = to a fixed is a hyperbola

Since ellipse and hyperbola have the same foci, they cut each other at right angles

intersect efficacy ab ⊥ ci a O fall thro
the ♀ point of intersect.



Defn the cal. of a curve
and def. the coords of the curve resp. by the gen.
= " of the second order.
To find the cal. of the curve

$$y = bx + \frac{c}{x}$$

of the cal. of the curve
when the eqn. for the coords become $\frac{0}{0}$
when that the = " resp. two // lines.

If a eq. be call. coords. then evidently
$Ax^2 + Bxy + Cy^2 = 0$. $xy = bx^2 + c$

Transpose $Ay^2 + Bxy + Cx^2 + f(hk) = 0$
into another of the form $Ay^2 + Bx^2 + f(hk) = 0$.
and then there is only one eqn of each quantity and
by subst this many can be effected
To find the area of the curve $ay^2 + dxy$ when $b^2 = 4ac$
is diff from Zero.

Prove that $y^2 - mxy + x^2 =$ An ellipse whose $a\left(\frac{1}{\phi} \neq 1\right)$

The = " bounded by $x^2 + y^2 = \left(A + mx + my\right)^2$
find the area. Let the first of the two.

Then that the degree of the $\pm \alpha$ in such a case
is always ~~is~~ the ~~same~~ order of curve
A st. line cannot meet a curve of the $n$ th.
order in more than $n$ points. may be
A curve of the $n^{th}$ order made to fulfill
$\frac{n(n+3)}{2}$ conditions.

Then how to find the points of the centres
of a curve

Ex. $x^2 + y^2 = (a + mx + ny)^2$.

Before a chain of a curve, - & when that is
must be of the $\frac{1}{2}n(n-1)$ order. of the proposed
curve be of the $n^{th}$ order

Then how to find the locus of the
middle points of a system of // chords
apply the method to curves of the $n$
th = $n$ th 2 curves by $x^3 + y^3 = 3axy$

$L\left(x^3 + y^3\right), a = x^2 y^2$

Then that a line inclined at $45^\circ$ to
the axis of $x$ is an axis of each of the

To find the distance betn. 2 pts, and the angle betn. the line joining them forms with the axes.

To find the eqn of a plane supposing it to be generated by a straight line which moves round a given fixed line &c

To find eqn of a st. line in space.

To consider that 2 planes may be ∥ to one another. also a st. line is ∥ to a plane, the projection of the line on the face of the plane upon any plane inclined at right angles to each other

To consider that a st. line and plane may be at right angles to each other

To find eqn of a st. line so drawn that it may make with given st. lines given angles

To find eqn of a st. line which

To find angle between 2 lines which having given both their eqns (2), the angles which they respectively make with the axes.

To find eqn of a plane thro' the origin, and ∥ to 2 given lines.

To find eqn of a plane which shall contain a given line and be ⊥ to a given plane.

To find eqn of generatively ~~and~~ a line moving along a line whose eqns are $x = mz + a$. $y = nz + b$. as it shall

Ld the area of the surf of a segment of a sphere.
volume of a segment of a sphere. ii
Spheroid
area of the surf of an oblate spheroid. deduce the area of the surf of a sphere
length of an arc of a conic (c).
volume of a figure generated by a plane moving
with its ends on a circle or ellipse — in perpendicular.
volume of a pyramid.
area of curve $dy^2 = a^2\sqrt{a^2 - x^2}$.
length of curve $a^3 y = x^4 + 6a^2 x^2$.
area of curve $y^2 = \frac{a^3}{2a - x}$.
Locus of the foot of a ⊥ dropped from a fixed pt. on the
tangent to ⊙ has for its $r = \rho + \alpha + \beta + \beta \sin\theta = \sqrt{a^2 + \rho + \alpha \cos\theta + \beta\theta} =$
$\text{area} = \frac{\pi}{2}\left(a^2 b^2 + \alpha^2 + \beta^2\right)$
arc of a cycloid rolls along a st. line bounds its area
the beginning and end of the motion. the area contd. between
the line and the curve which the locus of the vertex
$= \pi a^2 \left(1 + \frac{\pi}{6}\right)$

Find the condit. that 3 angles made by 4
lines thro' the orig. must satisfy.
Find = to a st. line passing thro' a given pt.
and ⊥ to a given plane.
Show that the angle between two planes = angle
between 2 st lines ⊥ to them
Investigate directly the condit. in order that two
st. lines whose = are given may be ⊥ to each other
The = $mm' + nn' + ll' =$
by $m = m'$ $n = n'$
Find the angle of intersect. of two planes whose = are
given
Find the angle of intersect. of a line and plane.
Find distance of given point from a given plane.
Find coords of the foot of the ⊥ dropped from a
given pt. upon a given plane.
Find the distance of a given point from a
given line
Find the coords of the foot of the perpendicular
dropped from a given pt. on a given line
Prove that the shortest dist. between 2 lines
meets them both at right angles
Find the = to a plane - viz. containing one given
line and ‖ to another.
Find = to a plane - viz. passes thro' a given pt. in

2 "to the directrix Hyd and then shall it be a catenary
conical asymptote.

1 = " to the ellipse Hyd and determine its form.

1 = " to H.i and determine the nature of the locus in
co-ordinates.

1 for the projection of a limiting line - " on any plane.
2 is a line.

" for area of the projection of any plane surface on a pl.

Express the distance from the origin and of 2 points from
one another by oblique coords.

The locus of a point whose distance from a fixed point
~~is to~~ bears to its distance from a fixed line
is a spheroid on a centrum Hyd whose
semi-axes $\frac{me}{1-m^2}$ & $\frac{me}{\sqrt{1-m^2}}$ e being the perpendicular
from the point upon the given line.

If an O roll on a straight line. its centre will describe
a circle whose = " -.

$$\frac{dy}{a} = \frac{y^2}{\sqrt{(a^2-y^2)(y^2-b^2)}}$$

the parab
the tangent & normal each make equal angles with the focal
radius, and a line drawn through parallel to the axis
which is the inclination of the axes when the two lines coincide
$y^2 - x^2 = 0$ by the eq. are + to each

Def a diameter, and shew that all diameters
of a parabola are straight lines // to axis

Find the eq. to a parab. referred to any diamet.
and the tangent at the extremity of diam as axis
the double ordinate of any diameter which the focus
is 4 times the focal distance of its ext.
∴ the tangent at the extremity of any chord
will intersect the diam. opposite the chord in
an ordinate.

Any parabolic segment is $\frac{2}{3}$ of the circ. whose
sides are the abscissa & ordinate.

Find the eq. to ellipse referred to
the form $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$

Shew that the ellipse has two foci whose
distance from each = $ae$, and the distance of directrix
from centre = $\frac{a}{e}$. Latus rectum = $2a(1-e^2)$

The Δ contained by 3 tangents to a parab is
double of Δ contained by the 3 chords joining
the points of contact.

Let the centre and axes of the curve, and then take the point $P$ on the curve whose co-ords are $x, y$

$$Ax^2 + By^2 = C$$

Find a Conjugate and then in that case the curve will be the two sides of the directrix

Find the eq to parab and trace the curve.

Show that $SP = PM$ and find the polar eq.

Find the eq to tgt and normal to any point.

and show that they are bisectors parallel & perp to the axis

(prove the subt = 2 abscissa, and subnorm = $\frac{1}{2}$ latus rectum)

$$SP = ST = SG.$$

The perp from the foci upon the tgt to the parab intersect the tgt in the line joining the parab -

$$SY^2 = S'P \times SA.$$

Find eq to tgt to parab in

$$y = mx + \frac{a}{m}.$$

Show that the locus of the intersection of the tgts to parab at the ends of a pair $(h, k)$

Find eq to normal under the form

$$y = mx - 2am - am^3$$

& thence the locus of the intersection of two normals at right angles to each other.

Find the locus of the intersection of two tgts to a parab making a constant angle.

$$\int \frac{1}{(1-x^2)^{\frac{3}{2}}} = (1-x^2)^{-\frac{3}{2}}$$

$$\int (1-x^2)^{-\frac{3}{2}} \cdot \int (1-x^2)^{-\frac{1}{2}}$$

$$P = x(1-x^2)^{-\frac{1}{2}} \quad \therefore \frac{dP}{x} = (1-x^2)^{-\frac{3}{2}}\{(1-x^2) + x^2$$

$$= (1-x^2)^{-\frac{3}{2}}$$

$$\therefore \int \frac{1}{(1-x^2)^{-\frac{3}{2}}} = \frac{x}{(1-x^2)^{\frac{1}{2}}}$$

$$\therefore \int \frac{dx^{-1}x}{(1-x^2)^{\frac{3}{2}}} = \frac{x dx^{-1}x}{(1-x^2)^{\frac{1}{2}}} - \int x$$

$$= \frac{x\,dx^{-1}x}{(1-x^2)^{\frac{1}{2}}} - \frac{1}{2}x^2 \text{ ——}$$

$$y = \frac{(4ay)^2}{4a} + c$$

$$4ay = 4ay^2 + 4ac$$

$$-\frac{4ac}{x} \quad \frac{4a}{x} \quad y \quad \therefore \quad y^2$$

Eliminate $a$ & $b$ from $(x-a)^2+(y-b)^2=c^2$.

$y = b \cos mx + a$.

$c$ from $\sqrt{x^2+y^2} = c\theta \cdot \tan^{-1}\frac{x}{y}$.

$a$ & $b$ from $2xy + ay^2 - bx^2 = 0$.

Explain why it does not admit of a denominator of the 2nd. order.

Integrate $2axy + ax^2 dy/dx = y^3 + 3xy^2 \cdot dy/dx$.

$x + y \cdot dy/dx = x\, d_x y - y$. $\sqrt{x^2+y^2} = c\theta^{\tan^{-1}\frac{y}{x}}$.

$(1-x^2) d_x y + 2xy = ax^2 + b$.

$(1+x^2) d_x y + xy + 1 = 0$.

$\sqrt{x} \cdot d_x y = 2y = a\sqrt{y}$.

$dx/y = \sin(mx + ny)$.

$d_x y + L + y\left(\frac{1}{2}\log\frac{R}{L}\right) + y^2 R = 0$.

$y = -\sqrt{\frac{L}{R}} \tan$ [illegible]

$y = -\sqrt{\frac{L}{R}} \tan \sqrt{RL}\, x$.

$L$ & $R$ are fns of $x$.

$y + \frac{1}{2} y^2 \cdot d^n x = 0.$

$d^2 y + 5 - \frac{2x^3}{x+x^4} - d_x y \cdot \frac{2+dx^3}{x^2+x^5} dy = 0.$

$y = x^2$. is a solut. find complete integral.

Ascertain the diff. between a total and partial diffl $= n$.

Integrate. $\frac{y \cdot dx - x\,dy}{y\sqrt{y^2 - x^2}}$. $\frac{2ay \cdot (y\,dx - x\,dy)}{(a^2+y^2)^{\frac{3}{2}} \sqrt{x^2+y^2}}$.

$\frac{x^{n-1}}{y^{m+1}} \cdot (my\,dx - nx\,dy)$

find the condit. for $L\,dx + A\,dy + R\,dz$ a dmtly a solutn of the form $f(x.y.z.c) = v$.

$(y+z)\,dz + (x+z)\,dy + (x+y)\,dz = 0.$

$2y \cdot dx + 2x\,dy + (ay + az^2)\,dz.$

$(y^2 + yz + z^2)\,dx + (x^2 + xz + z^2)\,dy + (x^2 + y^2 + xz) \cdot dz = 0$

find $= n$ Eqtn surface which has the property that the normal at any point passes thro' the centre of $\sqrt{}$ of the 4 whose ends are the coords of $x.y$ of the given point.

$x = n$ for a surface belonging to concave surfaces and a surface of revolut. about the axis

$(x-a) + y \cdot (y-b)\} dz = (x\,dx + y \cdot dy)(z-c)$ and that dis not integrable unless $a = b = 0$. Integrate dis the line

$\frac{\sqrt{a+bx^2}}{x^4}$ . $\frac{1}{(1+x)\sqrt{1-x^2}}$ . $\frac{1}{x\sqrt{a+bx^n}}$ . $\frac{1}{\sqrt[4]{a+bx^4}}$ . $\frac{1+cx^2}{(1-cx^2)}\sqrt{\frac{1}{1+bx^2+cx^4}}$ .

$\frac{x^5}{a^2+x^2}$ . $\frac{2x+3}{x^3+x^2-2x}$ . $\frac{x^2-2}{x^3+4x^2+4x}$ . $\frac{x^2-x+1}{x^3+x^2+x+1}$ .

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Resolve $\frac{1}{1-x^n}$ into partl. fracts. n by odd.

If ~~x^m~~ In the general $\frac{x^m}{1+x^n}$ , $\frac{x^m}{x^{2n}-2x^n\cos\theta+1}$ .

If $\frac{A}{x-a}$ is a partial fract in the resolution of $\frac{U}{V}$

$A = \frac{U}{\frac{dV}{dx}}_{x=a}$ .

In the resolut of $\frac{U}{Q(x-a)^n}$ the general term of the group of partial fracts corresponding to $(x-a)^n$ is $\frac{1}{\lfloor n}\cdot d^n\left(\frac{U}{Q}\right)$

Hence resolve $\frac{1}{(x-a)^n(x-b)^n}$

I should have thought it would have corrected on

the cent. of a ⊙ moving along a line $y = mx + c$
its rad increases as $x$ always does $\therefore$ $l$ to the
abscissa of pts cut. $f$ & the curve which
envelopes the ⊙s
the curve touched by the chord joining L and D
is an ⊙ is another ⊙ whose axes are $\frac{a}{2}$ & $\frac{b}{2}$.
the curve touched by the diam of a diam ⊙
which rolls along a st. line is a cycloid.
The curve touched by the base of a cycloid.
which rolls along a st line is

$$\frac{x}{a} = \left\{\left(\frac{y}{2}\right)^{\frac{2}{3}} + 2\right\}\sqrt{1 - \left(\frac{y}{2}\right)^{\frac{2}{3}}}.$$

Find the no. n tallthe tangents of a curve - the condtn being of algpc.
Find the condtn of a point of contrary flexure. and prove that the
curve is convex or concave &c as $y$ and $\frac{d^2y}{dx^2}$ have the same or diff. sign
Prove that 2 curves which have a contact of the nth order will touch
each accordg. as n is odd or even.

$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2$.

For a conic sect. $\rho = \frac{\text{Normal}^3}{(\text{Semi-latus rect.})^2}$.

For in an $\odot$ $= a - 2c + 3c\,d^2 I_4$. where $c = a - b$ & is small
Prove that at the points of greatest & least curv. the contact
of the $\odot$ of curv. is of the 3rd order.
Shew that a conic sect. may be described which shall have
contact of the 4th order with a curve at any point, and
if at any point it be a $\odot$ at the preceding & succeeding
points it will be $\odot$ and $\pm$.
Find the coordts of the cch. of curv. at a gen. pt. of $\odot$
Hence the no. of the evolute.
Find the no. of the asymptotes to the curve. $y^4 + 2ax^2y^2 - a^4 = 0$.
Prove that if $O$ be a point such that $AO + BO + CO$ is a min.
each of the $\angle$s at $O$ is $120°$.

$u = \frac{x}{a^2} + x^2$. Then $d_n u \left\{ (-1)^n \frac{1 \cdot 2 \cdots n}{a^{n+1}} \right\} \cos(n+1)\theta$ &c. $\theta^{n+1}$.

where $\cot\theta = \frac{x}{a}$.

Prove that the reciprocal of a semicircle with major axis // to the
the locus of the cch. of curv. of the $\odot$ at the point of contact
is $\rho = a \cos^2\theta$, $a =$ rad. of sem. $\odot$

$y^2 = \frac{b^2}{a}(a - x)$.

explan. in what cases. the true values of vanishg. fracts. cannot be found by successive diffs. of its numerator and denominator & shew how to find the true value in these cases.

Explain in what cases $\frac{dy}{dx}$ derived from $u = f(x,y) = 0$ will assume the form $\frac{0}{0}$. & how its values are to be detd.

$\frac{dy}{dx} = \frac{4y - 2bx}{3y - ax}$. find its value when $x = y = 0$.

Shew that $f(a)$ may be a max. or a min. for values of $x$ not furnished by the $= n$. $\frac{d}{dx} f(a) = 0$.

Investigate the condtn. that $f(x,y)$ may be a max. or min. where $x$ & $y$ are indepndt.

$x^4 + 2ax^2y - ay^3 = 0$. shewthat at the origin the values of $\frac{dy}{dx}$ are. $0$ and $\pm\sqrt{2}$.

Find the max and min values of $x$ and $y$.

Join the bases of 2 $\Delta$'s at the dem. of their heights, shew that in each case that the dem. of their areas may be a max. the $\Delta$s must be isosceles. & the dimen of the $\Delta$ is of the bases must be as the bases.

The $= n$ $p^2 = \frac{b^2 r}{2a - r}$. find the max & min values of $p$ and explan. the results.

Find the conjugate diam. of an $\odot$ whose dem. is a max. & dem a minimum.

Find the value of $\frac{a^3 \cos^2 x + \tan x}{x}$ when $x = 0$.

$\frac{\sqrt{x} - \sqrt{a} + \sqrt{x-a}}{\sqrt{a^2 - a^x}}$ when $x = a$.

If $a^3y = x^4$. dbl. pt at the origin is a pt. of inf.
$y = x^2 \pm x^{\frac{5}{2}}$. There is a cusp of the 2nd kind
at the origin
For $\frac{y}{a} = \sqrt{\frac{a}{x}} - 1$. when $x$ is less than $\frac{3a}{4}$.
If $\rho$ of the evol. of curve at $P$ is a $\odot$ and OX $\perp$ to
OP. meets CP in Q. prove that the rad of curv.
of the $\odot$ at $P = \frac{b^2}{a} \sec^3 \angle PQO$, and $\rho$ of the evolute
at $Q = 3QO$.
If $f(l.p) = 0$ be $= n$ to a spiral. then
$f\left(l, \frac{p^2}{p'}\right) = 0$ is the $= n$ to the locus of $p$.
Then how to draw a tangent to a curve. whose
polar rel. is defined by a rel. betn $\angle PAx$ and $r$.
$\int x \cdot \angle L + n + L = C$.
Then how to det. multiple and conjugate points.
$y^4 + 2ay^2x + x^4 - 2ax^3 = 0$ the origin is
a double point
$y^4 = x^5 + x^4 + 3x^2y^2 = 0$. then det the origin
is a conjugate point.
If $y^2 = \frac{x^3}{2a - x}$. there is a cusp of the 2nd kind
at the origin

prove that $\frac{1}{r^2} = \frac{1}{f_a} = a^2 + \left(\frac{da}{d\theta}\right)^2$.

$$r = \frac{a^3 \left(\frac{d\theta}{ds}\right)^3}{a + \frac{d^2 a}{d\theta^2}} \qquad \int \frac{d\theta}{ds} = \int \frac{1}{2} \rho^2 = \frac{1}{2} pfs$$

Also: the polar = $^n$ of the hyperbola.

$$\frac{\rho^2}{c^2} = \frac{a^2 - \rho^2}{a^2 - c^2}$$

Let the asymptotes be axes of rectangular coords. of $\frac{1}{a} = \frac{\theta^2}{\theta^2 - 1}$.
Prove the curve $\rho = \frac{\theta^2}{a + \sqrt{a^2 - b^2} \cdot \cos\theta}$. find the relation between $p$ & $f$.
Prove the involute of a $\odot$ is the $=^n$ to $p$ at $\rho$, and show that
the locus of $f$ is the spiral of Archimedes.
$\therefore a\alpha = ca^3 \frac{\theta}{3}$. find the relation between $p$ and $\rho$.
and show that the locus of $f$ is a $\odot$.
Find the $=^n$ to a curve which touches a series of curves
described after a given law.
The centre of curv. is the intersection of consecutive normals
at a point.
The ends of a straight line of given length move in
rectangular axes of coords. find the curve such that it
is always a tangent.
The evolute of an hypocycloid is another hypocycloid

the arc of a great ⊙ intercepted between
the equator, and the Zenith of a place measured
on a secondary to the

Reduce to an integrable form. Hopeless.

$$\frac{1}{(x+a)^m(x+b)^n} \qquad \frac{1}{(x^2-x)^2} \qquad \frac{1}{(x^2+a)^2(x^2+b)^2}$$

$$\frac{\alpha x+\beta}{(x+p)(x+q)} \cdot \frac{1}{\sqrt{a+bx+cx^2}} \qquad \frac{1}{\sqrt{a-x}-\sqrt{b+x}} \qquad dx = 2$$

$$\int \frac{1}{(x^{2n}+1)\sqrt{\{(x^{2n}+1)^{\frac{1}{n}}-x^2\}}} = \sin^{-1}\cdot (x^{-2n}+1)^{-\frac{1}{2n}}.$$

$$\int_0 \frac{\cos rx}{1+e\cos x} = \frac{\pi}{\sqrt{1-e^2}}\left\{\frac{\sqrt{1-e^2}-1}{e}\right\}^r.$$

Hence then $\int_0$ of $\frac{\cos rx}{(a+b\cos x)^n}$. $r$ and $n$ integers

$$\int_0 \frac{1}{x^2+c} = \frac{\pi}{2\sqrt{c}}. \quad \text{Hence deduce } \int_0 \frac{1}{(x^2+c)^n}$$

$$\int_0 t^{-t^2} = \frac{1}{2}\sqrt{\pi}.$$

$$\int_0 \sin mx = \frac{1}{m}.$$

Then how to integrate $\int x^n + p_1 x^{n-1} + p_2 x^{n-2} + \&c$

Ex. $(x^2+1)$.

2. Then how to integrate the products of consecutive terms of an arithmetic progn.

Also a fraction whose den. is a prod.t of consecutive terms.

Integrate $a^x \cdot \log^n_x \cdot \cos x$ & $\frac{1}{\cos^2 a \cos(a+1)^2} \cdot a^x \, dx$ &c.

Then how to resolve $x^3 + p_1 x^{n+} \&c$ into factors.

Resolve $x^4 - 3x^2$ into factors and integrate it

Also $\frac{1}{3x+1} \cdot \frac{1}{3x+4}$ ; $\frac{1}{x \cdot (x+1)(x+2)(x+3)}$

2. $\frac{x^2+6x+12}{x(x+1)(x+2)2^x} = \frac{x+3}{x \cdot (x+1) 2^{x-1}}$.

2. $\frac{1}{4x^2-9} = -\frac{1}{6} \cdot \frac{12x^2-12x-1}{(4x^2-1)(2x-3)}$.

$z = f\left(\frac{x-z}{y-z}\right)$. form the partial diffl eqn of the prob case of prob. it is the integral.

$p(1-z) + q(y-z) = 0$

$\frac{dz}{y} = \frac{a}{\sqrt{a^2-y^2-x^2}}$.

$px + qy + z = 0$ ; find the arbitrary fn so that when $y = ax^2$
$z = bx$.

$p(z+y) + q(z+x) = x+y$ ; $f\{(x+y+z)(z-x)^2, (x+y+z)(z-y)^2\} = 0$.

$(x+y)p + (y-x)q = z$. $z = (x^2+y^2)^{\frac{1}{2}} f\left\{\tan^{-1}\frac{x}{y} - \log\sqrt{x^2+y^2}\right\}$

$x^2 p + y^2 q = z^2$.

$(2az + cx)p + (2bz - cy)q = (ay + bx)c$

Ans. $z + ay - bx = f(z^2 - cxy)$

find a surface in wh the portion of the axes of y, z between the origin and tangent plane ∝ dist of the pt from the origin.

$\{z + \sqrt{x^2+y^2+z^2}\}\, x^{n-1} = f\left(\frac{y}{x}\right)$

find a surface wh cuts at right ∠s all surfaces confocal with the $= n$. $(ax^2 + by^2 + cz^2 = D$. when D assumes all possible values.

$\frac{z}{yc} = f\left(\frac{z}{x^c}\right)$.

integrate $d_x^2 z - d_y^2 z = (ax + by)$. $z = \frac{ax^3}{6} + \frac{bx^2 y}{2} + \varphi(y+x) + \psi(y-x)$.

$d_x^2 z - d_x d_y z + d_y^2 z = e^{(ax+by)}$

$z = \frac{e^{(ax+by)}}{(a-b)^2} + x\varphi(y+x) + \psi(y-x)$.

find a surface wh the area of any portion bears a constant ratio to the area of its projection on the plane of xy.

a right cone about the axis of z is a particular case.

$\sqrt{1+p^2+q^2} = \text{constant}$.

$\frac{4\pi a^3}{3} \cdot \Delta n \cdot \frac{a}{2}$.



$y = xt$.

$d_t V =$

$d_n \cdot d_t V = 2.$

$d_1 \cdot d_t V = x 2.$

$= x \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}.$

$y = xt.$

$= x \sqrt{1 - \frac{x^2}{a^2} - \frac{x^2 t^2}{b^2}}$

$= \frac{x^2}{b} \sqrt{\frac{b^2}{x^2} - \frac{b^2}{a^2} - t^2}.$

$= \frac{x^2}{b} \left\{ \frac{t}{2} \sqrt{\quad} + \frac{1}{2} \left( \frac{b^2}{x^2} - \frac{b^2}{a^2} \right)^{2} \Delta t^{-1} \frac{t}{\sqrt{\frac{b^2}{x^2} - \frac{b^2}{a^2}}} \right.$

$d_x y = f(ax+by+c)$. Separable variables.

$dy = \dfrac{3xy^2 - x^3}{1+6y^2-3x^2y}$. Exact.

$d_x y = xy^2 + x$.

$x d_x y - y = \sqrt{x^2+y^2}$.

$\dfrac{x^2+y^2}{a+x} + y d_x y = 0$. $\quad d_x y + y = a$

$+ d_x y + by^2 = ax^m$. Separate the variables if $m = \begin{matrix} -2 \\ -4 \end{matrix}$

$\dfrac{x dy - y}{\sqrt{1-(dy)^2}} = f(x \cdot y)$ Separable variables.

$M + N dy = 0$ be homogeneous. $\dfrac{1}{Mx+Ny}$. renders

Exact. Hence integrate. $\dfrac{xy+y^2}{xy-x^2} + d_x y = 0$.

Investigate a factor which will make $d_x y + P y - R = 0$ integrable

$y d_x y + y + \dfrac{1}{cx} = 0$. becomes integrable by the

factor $\dfrac{1}{x}\left\{e^{-\frac{c}{2}(x+y)^2}\right\}$

$d_x y = \dfrac{y(n+cx)}{y+a+bx+cx^2}$ Separate the variables by

substitution $x = z$.

The trajectory of a st. line $y = cx$ cuts on $= n$

a log. $\dfrac{\sqrt{x^2+y^2}}{C} = \tan^{-1}\dfrac{y}{x}$.

Find the diff. $= n$ to the trajectory of a curve

of the $2^{nd}$ order.

Find the curve the locus of the orthogonal

intersect is a curve of the 2nd order.

$u = \log.(a^2 b^2) - \log b^2$

$\frac{du}{dx} = \frac{1}{b^2}{}^u = \frac{a^2 b^2}{b^2}$

$\therefore \frac{du}{dx} \frac{1}{b^2}{}^u = \frac{2x}{b^2}$

$\therefore du = \frac{2x}{b^2} \cdot \frac{b^2}{x^2 b^2}$

$= \frac{2x}{x^2 b^2}$

$d_x^2 y + a^2 y = 0$.

$d_x^2 y = ax + by$.

$L.d_x^1 y + \frac{1}{2}\cdot\frac{a}{x} L dx y + cy = c$.

$L$ a fn of $x$

$y L.d_x^2 y + d_x y = \frac{y}{x}$.

$d_x^2 y - (x+a) d_x y - \frac{x}{y}(d_x y)^2 = 0$.

$(1+x^2) d_x^2 y + (d_x y)^2 + 1 = x$

$x.d_x^2 y - d_x y = x^2$.

$d_x^2 y + (e^x - 1) d_x y = e^{2x}$.

Find the circum rad. $\rho$. it is a fn of the abscissa.

$\rho$ = normal. as the curve concave or convex.

$\rho$ varies as the cube of the normal.

$\rho$. varies as the sec cube of the $\angle$ which the rad. vector makes with the curve

Reduce. $E(d_x^2 y . d_x y : y) = 0$ to $= n$ of the first order.

$d_{xy}^2 y . d_x^2 y = a$

$x(a+bx) d_x^2 y + (c+ex) dxy = a$

$y^3 . d_{xy}^3 y = c$

Integrate $y d_x^2 y + (d_x y)^2 = 0$.

$y d_x^2 y + \sqrt{1+(d_x y)^2} = 0$.

$d_x^2 y = f(y)$.

If $u$ be a particular integral of the $= n$. $d_x^2 y + P d_x y + R = 0$ the complete integral is $u \int \frac{e^{\int P dx}}{u^2}$.

Explain the term singular solut. of a diffl = n.
Then how to obtn all the singular solutions from the
general integral
Hav. given a solut of a diffl = n. then whether it is
included in the complete integral or not.
Then how to get the singular solut. from the
diffl = n. without knowing the complete integral.
Every factor proper to make a proposed = n integrable
is made impossible by the singular solut.

$y^2 - 2xyp + (1+x^2)p^2 = 1$. $y^2 = 1+x^2$ is the sing. solut.

$(xp-y)(xp-2y) + x^3 = 0$. $y^2 - 4x^3 = 0$ is the solut.

diffl = n. $yp^2 - 2xp - y = 0$. $x^2+y^2=0$ is a singular
solut. $y^2 = 2x+1$ a particular integral.
Find the curve in wh. the length of tangent intercepted
by the axes of x is measurable.

the product of this $\perp$ is upon the
tangent from 2 given pts is invariable. from a point
such that by drawing 2 lines to
2 given pts at a given angle the intercepted pts
on the curve wh $\perp$ is y. $\alpha$. we may always have
$\alpha^2\beta = \text{const}^2 y$
Find singular solutions of.
$dy = \frac{ydx}{x-dy}$; $(y-dy)^2 = b^2 + a^2(dy)^2$.
$(x+1)^2 = 4y$.

$x^2 \cdot d_x^2 y - x d_x y + y = a \log x$

$x^2 \cdot d_x^2 y - 2x \cdot d_x y + 2y = 0$.

Laplace a.7. in Cases of t from the $= \infty$.

$$d_t x + y + 5x = e^{-t}$$

$$d_t y - x + 3y = e^{2t}.$$

Solve. $d_t^2 x + d_t^2 y + x + 2y = 2$.

$$4 d_t^2 x + d_t^2 y - 3x + y = e^{-t}.$$

Apply $= \infty$. $y = y_{x=a} + \frac{x-a}{1} \cdot d_{x=a} y + \alpha - \omega$

integrate $d_x^2 y = \int \cdot (x y \cdot d_x y)$.

Ex. $y - x d_x y + \frac{1}{2} x^2 \cdot d_x^2 y = 0$.

$$d_x^2 y + \frac{5 + 2x^3}{x + x^4} d_x y - \frac{2 + x^3}{x^2 + x^5} 6y = 0$$

$y = x^2$ is a solution for the complete integral.

Reduce the linear eqn of the nth. order to one with const coeffts

$$-D^2y - 3Dxy + 2y = 0$$

Integrate the linear eqn. of the nth. order with variable coeffts.

$$-D^2y + 2Dxy + y = x^3.$$

$$-D^2y + n^2y = 0 \quad \text{solved when } x=0 \text{ we may have } y=1 \; Dy=0.$$

$$-D^2y + \frac{1}{x}Dy - \frac{1}{x^2}y = 1 \quad \text{solved when } x=1 \text{ we may have } y=a \cdot Dy = b.$$

$$-D^2y + n^2y + ax + b = 0.$$

$$-D^2y + 2nDy + n^2y = 0.$$

$$-a^2D^4y - D^2y = 0.$$

$$-D^4y - 4D^3y + 6D^2y - 4Dy + y = 0$$

$$-D^4y + 2n^2D^2y + n^4y = 0$$

$$-D^2y + n^2y + b\cos nx + a = 0.$$

$$-D^2y + n^2y = a\cos nx + a.$$

$$-D^2y - a^2y = e^x.$$

$$-(a+bx)^2D^2y + A(a+bx)Dy + By = f(x)$$

Reduce to linear eqn with constant coeffts.

$$x^3D^2y = (y - xDy)^2 \qquad y = x\log\left(\frac{C+C_1x}{x}\right).$$

$$D^4y - a^4y = x^3. \qquad y = -\frac{x^3}{a^4} + Ce^{ax} + C_1e^{-ax} + C_2\cos ax + C_3.$$

$$(D_x - a)^r y = e^{ax}.$$

$$\therefore y = \frac{e^{ax}}{(a-a)^r} + \{C_1 + C_2 x + C_3 x^2 \cdots\} e^{ax}.$$

$$(D_x - a)^r y = e^{mx}.$$

$$\therefore y = \frac{e^{mx}}{(m-a)^r} + C_1 + \cdots\cdots\} e^{ax}$$

$$= \frac{e^{mx} - e^{ax}}{(m-a)^r} + C' \cdots.$$

$$= \frac{x \cdot e^{mx}}{r\,(m-a)^{r-1}}$$

$$= \frac{x^r \cdot e^{ax}}{\lfloor r} + (C' + C_1 x \cdots) e^{ax}.$$

$$D_x^2 y + n^2 y = \cos mx.$$

$$(D_x^2 + n^2) y = \cos mx$$

$$\therefore y = \frac{\cos mx}{-m^2 + n^2} + A \cos nx + B \sin nx.$$

$$(D_x^2 + n^2) y = \cos mx.$$

$$\therefore y = \frac{\cos mx - \cos nx}{n^2 - m^2} + A \cos nx.$$

$$= -\frac{x \sin mx}{2n} = \frac{x \sin mx}{2n} + A.$$

If $p = f(\varphi)$ be the relation between the radius vector & $\perp$ to the tangent. Then locus of the pole when the curve rolls along a st. line is the plane is $y = f(y, d_x y)$.
If the curve be a $\bigcirc$ the locus of the focus is the common catenary.
The locus of Cusp- if rolling along a st line is $d_x = \frac{y^2}{\sqrt{a^2 - y^2}\sqrt{y^2 - b^2}}$
Integrate $y = xp + f(p)$. Ex. $\frac{x + y d_x y - c}{x - c} = \frac{(x + y d_x y)^2}{x^2 + y^2}$.
Integrate $y = x f(p) + \varphi(p)$.
Find a curve whose subnormal is constant.
$= \propto x$.

(normal)$^2 \propto$ abscissa + subnormal

Find a curve in wh. the normal bisects the $\angle$. $y^2(1+p^2) = a^2(x+py)$.
between the focal distances.
$2c$ be the distances of the 2 foci & $y$ a mean & the middle point the $= n$ to the curves.

$$(1-p^2)xy = p(x^2 + y^2 - c^2).$$

Integrate. $(axy)(d_x y)^2 + (x^2 - ay^2 - b)\, d_x y - xy = 0$.
Assume $d_x y = \frac{x}{y} \cdot a$.
$d_x y + \frac{1}{2} d_x^2 L = y^2$.
$y + \frac{dL}{x} + \frac{1}{p_n} \cdot \frac{1}{\int x \frac{1}{p_n}} = 0$. let $y + \frac{dL}{L} = \frac{1}{2}$.

$\frac{1}{x(a+bx)^{\frac{2}{3}}}$ $\quad$ $\frac{x}{(x^3+a^3)^{\frac{2}{3}}}$ $\quad$ $\frac{x}{\sqrt{a^2x+b^2x^2}}$.

$\int_0^\infty \frac{\cos rx}{q^2+x^2}\,dx = \frac{\pi}{2q}\cdot$ Hence deduce $\int_0^\infty x^p \frac{\cos rx}{q^2+x^2}\,dx$.

If $p$ be the $\perp$ upon the tangent from an assumed point in the axis and $\theta$ the $\angle$ which it makes with the axis. $r$ the length of the tangent which intercepts and $\phi$ the corresponding arc of curve then $d_\theta(s-r) = p$.

Hence &c. the length of arcs of the diff. of whose lengths can be expressed algebraically.

Find a series for the length of an elliptic quad. whose ecc. is not greater than $\frac{1}{\sqrt{2}}$.

Find the area of the whole surface of a spheroid.

Find the vol. of any segment of a spheroid.

Shew how to approx. to the area of a curve by the method of ordinates.

Find the area of the surf. of the angle of any cone cut off by a plane thro' the centre of its base

Find the area of the node of the curve

$$y^3 + 2axy + x^3 = 0.$$

Find the area of a sector of a spiral whose $= n$ is $\rho = a\sec\theta + b$.

If the $= n$ be a spiral be $\rho = \frac{a}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$

the area $= \frac{\pi}{n}\left\{\frac{a}{b}\right\}^2$

Prove that term of 3rd order multiplied into coeff=$a^2$ from $\frac{1}{r}$ to $S$ [illegible] 2nd order

$$\frac{d^2}{d\theta^2}S + S + \frac{3}{2}m^2k\sin(\theta-\gamma) - \frac{3}{2}mk\sin(2-2m-q)\theta-2\gamma-q.$$

[illegible] the value of $S$ and interpret the result.

Explain why the motion of the node is obtained by the 2nd approx: [illegible] than the motion of the apse.

Given the value of $\theta$ the mean long.

$$pt - \frac{2e-\frac{1}{4}e^3}{2}\sin(pt-\alpha) + \frac{5}{4}e^2\sin 2(pt-\alpha) + \frac{15}{4}me\sin(2-2m-c)pt-2\beta+\alpha$$

$$-3me\sin(mpt+\beta-\zeta)$$ find from this the mean velocity of the apse in any revolut. then that the apse line progresses in $d\,g\,g\,g\,g$. and regress in quad. find also the periodic time of the Moon in any revolut. and shew how it depends on the position of the sun.

$\int \frac{p+qx^2+rx^4}{1-c^2x^4} \cdot \frac{1}{\sqrt{1+px^2+c^2x^4}}$ .

Assume the rational part $= A \cdot \frac{1+cx^2}{1-cx^2} + B \cdot \frac{1-cx^2}{1+cx^2}$.

Let $A \& B$.

Ex. (1) $\int \frac{1+c^2x^4}{1-c^2x^4} \cdot \frac{1}{\sqrt{1+px^2+c^2x^4}}$ (2) $\int \frac{x^2}{1-c^2x^4} \cdot \frac{1}{\sqrt{1+px^2+c^2x^4}}$ (3) $\int \frac{\sqrt{1+px^2+c^2x^4}}{1-c^2x^4}$

1. $\int \frac{b+cx}{x^3+a^3}$ ; $\frac{1}{x^4-4x+3}$.

Integrate without substitut. $x^{m-1} \cdot (a+bx^n)^{\frac{p}{q}}$. $\frac{m}{n} = +$ve integer.

$\frac{m}{n} + \frac{p}{q} = -$ve integer.

$\frac{1}{(a+bx^n)(a+2bx^n)^{\frac{1}{2n}}}$ . Let $a+2bx^n = \left(\frac{x}{z}\right)^{2n}$. integral $= \int_z \cdot \frac{1}{a+b^{-1}z^{2n}}$.

If $a+2bx^n = z^{-2n}$. $\int \frac{x^{n-1}}{x\,(a+bx^n)(a+2bx^n)^{\frac{1}{2n}}} = -\frac{2}{b} \cdot \int_z \frac{1}{1+az^{2n}}$.

Rationalize. $\frac{\alpha+\beta x+\gamma x^2}{p+qx+rx^2} \cdot \frac{1}{\sqrt{a+bx+cx^2}}$

1. $\frac{x^{m-1}}{(a+bx^n)^{\frac{m}{n}}}$ . rationalize and integrate. $\frac{x}{(1+x^2)^{\frac{2}{3}}}$.

$D_x^2 y = a \cos mx + b \cos nx.$

$D_x^2 y + n^2 y = a \cos mx + b \cos nx.$

$D_x^2 y + 4y = 1 + \sin^2 x + \cos^2 x.$

write down the proper assumption for solving a solution shall show the forces on the moon and find the value of $T$. Show that this force always tends towards the nearer body.

Check the diff$^{ls}$ for determining $r$. the tangent of the moon latitude. in solving this $=^n$ state what difficulty arises when the result is put into such approx$^{ns}$ is used in the above form and how the difficulty may be cured.

Explain why it is necessary to retain in the diff$^{ls}$ $=^n$ some terms of the 2nd order to obtain a solut$^n$ which is correct to the 2nd order.

Supposing the tangential force the only disturbing force on the moon, find the value of $a$ to the 2nd order and interpret it. Suppose $\theta$ of the 2nd order.

Prove $\int_0^\infty e^{-ax^2} \cos cx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\, e^{-\frac{c}{4a}}$.

Prove $\int_0^\infty e^{-ax^2} \left(e^{mx} + e^{-mx}\right) = \sqrt{\frac{\pi}{a}}\, e^{\frac{m^2}{4a}}$.

Prove $\int_0^\infty \frac{x^{m-1}}{1+x} = \frac{\pi}{\sin m\pi}$ when $m > 0 < 1$

$$\int_2^\infty \frac{\pi}{2} \cdot (\log 2)^{2m-1} = \frac{\pi}{2 \sin m\pi}.$$

Prove $\int_0^1 \frac{x^{p-1}}{(1-x)} = p\,\frac{\pi}{\sin p\pi}$ Prove $\int_0^1 \frac{x^{p-1}}{(1+ax)(1-x)^p} = \frac{\pi}{(1+a)^p \sin p\pi}$.

Define a recurring series and prove the rule may be generally applied into 2 or more geometl. Progs.

2nd General term of $1+4+18+88+354$ &c.

Find x terms of the series of which every term beginning with the 3rd = sum of the 2 preceding.

Explain when a series is said to be convergent or divergent. $a + ax + ax^2 +$ &c is convergent divergent according as x is $< 1$ or $> 1$.

If the limit $\left(\frac{u_{n+1}}{u_n}\right)$ when $n = \infty$ be $< 1$

the series $u_0 + u_1 +$ &c is convergent and divergent in the contrary case.

$\int_0^\infty e^{-at^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}$. Hence $\int_0^\infty t^{2n} e^{-at^2} = \frac{1\cdot 3\cdot 5\cdots 2n-1}{(2a)^n}\cdot\frac{1}{2}\sqrt{\frac{\pi}{a}}$.

$\int_0^\infty t^{2n+1} e^{-at^2} = \frac{1\cdot 2\cdots n}{2a^{n+1}}$

$\int e^{ax}\sin(mx+\alpha) = \left(\frac{e^{ax}}{a+c\cos\theta}\right)\sin(mx+\alpha-\theta)$. $\tan\theta = \frac{m}{a}$.

$\int_r e^{ax}\sin(mx+\alpha) \quad \frac{e^{ax}}{(a+c\cos\theta)^r}\sin(mx+\alpha-r\theta)$

$\int e^{ax} x^n \sin(mx+\alpha) = \left(\frac{e^{ax}x^n}{a+c\cos\theta}\right)\sin(mx+\alpha-\theta)$

$-\frac{e^{ax}nx^{n-1}}{(a+c\cos\theta)^2}\sin(mx+\alpha-2\theta)\ c^{2n+1}\cos$

Find $=^{ns}$ of cond$^{ns}$ that 2 lines may be at right $\angle^s$ to one another or that a line and plane may be at right angle to each other.

the axes oblique or rectangular.

Find $=^{ns}$ of 3 planes passing thro' the same point find the cond$^{ns}$ that they may be in one plane.

Def. the point of intersect$^{n}$ of a plane and $\perp$ dropped upon it from a given pt. and the length of $\perp^{r}$.

Def. the point of intersect$^{n}$ of a line and a $\perp$ let fall upon it from a given pt. and the length of $\perp^{r}$.

Find the $\perp^{r}$ of intersect$^{n}$ of 2 planes whose $=^{ns}$ are given and also of a line and plane.



$$\int_0^\infty \frac{x^{m-1} + x^{n-m-1}}{(1+x^n)} = \frac{\pi}{n} \operatorname{cosec} \frac{m\pi}{n}.$$

Prove $\int_0^1 \frac{x^{m-1} - x^{n-m-1}}{(1+x^n)\log x} = \log \tan \frac{m\pi}{2n}.$

$$\int_0^\infty \frac{\xi^{ax} - \xi^{-ax}}{\xi^{\pi x} - \xi^{-\pi x}} \frac{dx}{x} \sin mx = \frac{1}{2} \cdot \frac{\xi^m - \xi^{-m}}{\xi^m + 2\cos\alpha + \xi^{-m}}.$$

Prove $\int_0^\infty \frac{\xi^{ax} - \xi^{-ax}}{\xi^{\pi x} - \xi^{-\pi x}} \frac{dx}{x} \sin mx = \tan^{-1}\left(\frac{\xi - 1}{\xi + 1}\right) \tan \frac{\alpha}{2}$

$$\int_0^\infty \frac{\xi^{ax} - \xi^{-ax}}{\xi^{\pi x} - \xi^{-\pi x}} \frac{dx}{x} \cos mx = \frac{\sin\alpha}{\xi^m + 2\cos\alpha + \xi^{-m}}.$$

$\int_0^\infty$ = "Cont. only 2 variables, one representing cylindrical surface // which axis whose coords it does not involve surface // $\int_0^1$ = "$\int_0^\infty$ contains Hord.; & the value of the chords // plane of yz. Prove that it can have an infinite no. of st. lines entirely consisting with the surface.

$\int_0^1$ = "to Hei (Ord. at them that it has plane asymptotes and can have an infinite no. of st. lines consisting with the surface.

Find the length of projection of a limited line upon another line.

Find the length of perp. from the origin & pts from one another in terms of oblique coords by.

$\int_0^1$ = "to the surface generated by a st. line which constantly passes thro' 2 fixed lines.

$\int_0^1$ the locus of a pt. whose dist. from origin (or dist. measured to the plane of yz. from a given line

or from one system of coords to another one oblique & rectangular & the oblique.



$$\int_0^1 \frac{x^{n-1} + x^{n-n-1}}{1+x^n} = \frac{\pi}{n} \operatorname{cosec} \frac{n\pi}{n}$$

Integrate each side to $n$.

$$\int_0^1 \frac{x\left(x^n + x^{n-n}\right)}{1+x^n} = \frac{\pi}{n} \operatorname{cosec} \frac{n\pi}{n}$$

$$\int_0^1 \frac{x\left(x^n + x^{n-n}\right)}{(1+x^n)\log x} = \frac{\pi}{n} \int_n \frac{1}{\sin \frac{n\pi}{n}}$$

$$= \frac{\pi}{n} \cdot \frac{1}{\frac{\pi}{n}} \log \tan \frac{n\pi}{2n}$$

$$= \log\left(\tan \frac{n\pi}{2n}\right)$$

Spirals proved the following.

Let $SPL = \theta$.

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2.$$

$$\left(\frac{ds}{d\theta}\right)^2 = \rho^2 + \left(\frac{d\rho}{d\theta}\right)^2.$$

$$r = \frac{u^3 \left(\frac{ds}{d\theta}\right)^3}{u + \frac{d^2u}{d\theta^2}}$$

$$\frac{d}{d\theta} HSP = \frac{1}{2}\rho^2 = \frac{1}{2} p \frac{ds}{d\theta}$$

expresses the rad of curv. in terms of $p$ & $\rho$.
Let the asymp. of the hyp. spiral

$$\rho_a = \frac{\theta^2}{\theta^2 - 1}$$

obt. the polar = in both cycloid

$$\frac{p^2}{c^2} = \frac{a^2 - \rho^2}{a^2 - c^2}.$$ Hence them obtain evolutes

a dim. epicycloid.

The curve $\rho = \frac{b^2}{a + \sqrt{a^2 - b^2}\cos\theta}$.

then obt $p^2 = \frac{b^2 \rho}{2a - \rho}$

$\rho_a = \frac{\sec\frac{\theta}{2}}{\cos\theta}\left(\sec\frac{\theta}{3}\right)^3$ & the relat. betw $p$ and $\rho$ is $p^3 = \rho^2 a$.
and prove the locus of $y$ in the common
& relat. betw $\rho$ & $\theta$ — the involute of a ⊙ as evolute

$a = x^4 - 8x^3 + 24x^2 - 32x$ find a rel. max or min if $x = 2$.

The sum of the edges of a rect. parallelepiped is a const and one edge is double of the other. Show that the remng. edge must $= \frac{a}{3}$ in order that the solid may be a max.

Def. osc. 2 curves are said to have a contact of the $n^{th}$ order. Prove that two curves wh. have a contact of the $n^{th}$ order while touch on ord. each other as $n$ is odd or even.

Def. the circle of curv. of fixed rad — the coord. of cent.

Write down rad of curv. in a polar $\rho = \frac{(S\rho)^{\frac{3}{2}}}{S\rho\rho - \rho^2 - 2(S\rho)^2}$.

Any conic sect. rad of curv = $\frac{(\text{normal})^3 \left(\frac{I}{2}\right)}{\left(\frac{I_1}{2}\right)^2}$.

The rad of curv = $\frac{b^2}{a(1 - e^2 \sin^2 \alpha)^{\frac{3}{2}}}$

where $\alpha$ is the angle wh. normal makes with major axis.

If $(a - b) = c$ $\rho = (a - 2c + 3c \sin^2 \alpha)$ nearly.

In the cycloid the rad of curv = 2 normal.

Catenary — $\frac{y}{a} = \frac{1}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$

rad of curv = normal.

Prove that rad of curv = $\frac{1}{\sqrt{(\sigma_s^2 x)^2 + (\sigma_s^2 y)^2}}$

it height of the least cone that can circum. a given
sphere is 2 diam.
circle.
Such sector must be taken out of a given ~~concent.~~
order that the rem. may form a cone of the gr. volume.
The angle of the sec. is $2\pi\left(1-\sqrt{\frac{2}{3}}\right)$

$u = 3x^5 - 125x^3 + 2160x$ for its greatest max.

when $x = 3$.

$$\int \frac{1}{\sqrt{(1-x^2)(x+2)}} = \sin^{-1}\frac{\sqrt{x+2}}{3}$$

$$\int \frac{1}{(1+x)\sqrt{1+x-x^2}} = \sin^{-1}\frac{3x+1}{(x+1)\sqrt{5}}$$

$$\int \frac{\sqrt{ax+bx^3}}{x}\,dx = -\frac{2\sqrt{3}}{3}\log\left\{\sqrt{\frac{a}{x^3}}+\sqrt{b+\frac{a}{x^3}}\right\} + \frac{2}{3}(a+bx^3)$$

$$\int (x^2+a)\sqrt{a^2+b}\,dx = \frac{x}{4}(x^2+b)^{\frac{3}{2}} + \left(a-\frac{1}{4}b\right)\left\{\frac{x}{2}\sqrt{x^2+b} + \frac{b}{2}\log(x+\sqrt{x^2+b})\right\}$$

$$\int \frac{1}{(x+a)\sqrt{x^2+a^2}} = \left(\frac{x}{\sqrt{a^2+a}}\right) - \left\{\frac{2}{\sqrt{a^2+a}}\right\}\tan^{-1}\sqrt{\frac{a-ax}{a+ax}}$$

$$\int \frac{x^2-2}{x^3+6x^2+4x}\,dx \qquad \int \frac{x^2-x+1}{x^3+x^2+x+1}\,dx$$

$$\int \frac{x^3+a^3}{(x+a)^3}\,dx \qquad \int \frac{x^2+1}{(x^2+3)(x^2+5)}\,dx \qquad \int \frac{1-x\cos\alpha}{1-2x\cos\alpha+x^2}\,dx$$

$$\int \frac{1}{x\sqrt{a+bx^2}}$$

$(a-x)(b-x)^{\frac{m}{n}}$.

$$\pm\frac{cx+bx^3}{(a+bx^2)^n} \qquad \pm\frac{1}{\sqrt{a-x}} \qquad \pm\frac{x^{n-1}}{\sqrt{a+bx^n}}.$$

$$\frac{1}{x^2+a^2)^{\frac{3}{2}}}$$

$\int x^m (a+bx^n)^{\frac{p}{q}} x^{m-1}$. is such a form that only $\frac{m}{n}$ is a fract. number or $\frac{m}{n}+\frac{p}{q}$ a neg. it is integrable

1. the integ. after a certain form.

— circular forms.

1. $\int a^2 \pm x^2 \cdot dx$ $\quad \int 2ax \pm x^2 \cdot dx \cdot \sqrt{a^2-x^2}$

$$\frac{1}{a^2+bx+cx^2} \qquad \frac{1}{x\sqrt{a+bx+cx^2}} \qquad \frac{1}{x^2\sqrt{a+bx+cx^2}}.$$

2. $\frac{1}{(c+ex)\sqrt{a+bx}}$ $\quad \frac{1}{(c+ex)^2\sqrt{a+bx}}$ $\quad \frac{x}{(c+ex^2)\sqrt{a+bx}}$.

3. $\frac{1}{(a+bx)(c+ex)}$. $\quad \frac{1}{(c+ex)\sqrt{ax+bx^2}}$ $\quad \frac{1}{p+qx\sqrt{a+bx+cx^2}}$ $\quad \frac{x}{(c+ex^2)\sqrt{a+bx}}$.

$$\frac{p+qx^2}{(a^4-x^4)\sqrt{a+bx^2}} \qquad \frac{(p+qx)\sqrt{a+bx^2}}{c+ex^2} \qquad \frac{1}{(px+qx^2)\sqrt{a+bx+cx^2}}.$$

$$\frac{x^2+ac}{x\sqrt{(a^2-x^2)(x^2-c^2)}}.$$

every inteq. an arbitrary const. must be added. the value of which will be known if the value of the integ. corresponding to a given value of x be known.

negate the integ. of $x^{m-1}$ and deb. the const. suppose the value of the integ. = 0 when $x = a$.

$a x^{m-1} d_x u$. $u$ being a fn of $x$.

$\frac{d_x u}{\sqrt{u}}$

having given $\int x^m = \frac{x^{m+1}}{m+1} + C$ deduce the result $\int \frac{1}{x} = \log_e x + C$.

Integrate $\frac{m d_x u}{u}$. when $u$ is a rat. fn of $x$.

$\therefore u = \int_x (u . d_x x)$. $u$ a fn of $x$ and $x$ a fn of $z$.

Know how to integ. $(a+bx^n)^{\frac{p}{q}} x^{m-1}$. when $\frac{m}{n}$ a posit. or $\frac{m}{n} + \frac{p}{q}$ a negy. int.

Integ..

$u = (2x^3 + 3x^2)^4 (x^2+x)$, $\frac{b+2cx}{\sqrt{a+bx+cx^2}}$ $\frac{1}{\sqrt[3]{a-x}}$, $\frac{1}{a+bx}$

$\frac{1}{bx+cx^2}$, $\frac{2-4x}{x^2-x-2}$, $\frac{5+2x^3}{x}$, $a^2\sqrt{a+x}$

$\frac{1}{x^3\sqrt{2ax-x^2}}$, $(a+(a-x)\sqrt{b-x}$, $(x^2+1)\sqrt{x^2+4}$.

$\frac{1}{x+x^{\frac{2}{3}}}$. $x^3\left(\sqrt{a+bx^2}\right)^{\frac{2}{3}}$. $\frac{m+n\sqrt{x^{-n}}}{x+x^{-n+1}}$

$\frac{6x^6 - 3x^5 + 2x^3 + 4x - 2}{7x^4 - x^3}$.

$= (a-b+b-x)\sqrt{b-x} = (a-b)\sqrt{b-x} + (b-x)^{\frac{3}{2}}$.

Applies when $n$ is much by taking an integral between limits.

$\int_0^1 \sqrt{a^2-x^2}\,dx \;/\; \int_0^1 x^{n-1} \;/\; \int_0^\infty e^{-ax}\sin nx \;/\; \int_0^\infty \frac{\sin nx}{x} = \frac{1}{n}$.

$\left(\int_0^{\frac{\pi}{2}} \sin x\right)^n \;/\; \int_0^\infty \frac{x^n}{e^{ax}} = \frac{1\cdot 2\cdot \lfloor n}{a^{n-1}}$. $\int_0^\infty \left(1+\frac{x^2}{n}\right)^{-n} = \frac{\pi\sqrt{n}}{2^n} \frac{1\cdot 3\cdot (2n-1)}{2\cdot 4\cdot 6\cdot (2n)}$

when when becomes when $n$ infinite.

$u = C + xu + \frac{x^2}{1\cdot 2}\frac{du}{dx} + \frac{x^3}{1\cdot 2\cdot 3}\,d$

$v_x = v_{x=a}\,(x-a) + \frac{dv}{dx}_{x=a}\frac{(x-a)^2}{1\cdot 2} + \frac{d^2v}{dx^2}_{x=a}\frac{(x-a)^3}{1\cdot 2\cdot 3}$ &c.

$\frac{1}{\sqrt{a+bx}} = \frac{2}{b}\left\{\sqrt{a+b}\cdot \sqrt{a}\right\}$. $\int_0^1 \frac{1}{\sqrt{a+bx+cx^2}} = \frac{1}{\sqrt{c}}\cdot\left\{\frac{2c+b+2\sqrt{c}\sqrt{a}}{b+2\sqrt{ac}}\right.$

and when how the former result this contd in

$\frac{1}{a+b\cos x} = \frac{\pi}{\sqrt{a^2-b^2}}$ $\int_0^\pi \frac{1}{(a+b\cos x)^n} = \left(\frac{\pi a}{(b^2)}\right)^{\frac{3}{2}}$.

$\frac{\cos nx}{1+b\cos x} = \frac{\pi}{\sqrt{1-e^2}}\left\{\frac{\sqrt{1-e^2}-1}{e}\right\}^n$

$\frac{1}{n^2+c}$ . $\int x$ $\frac{1}{\sqrt{2x-x^2}}$.

expand by $\log(1+x)$ $\tan^{-1}x$. $\sin^{-1}x$.

= $n$ of $\phi$, must have equal roots or not a condition
for $\phi$, or $f'\phi$, have a comm. meas.
Then how to find the commensurable root when $= 2$.
The root of an algebraical $= n$ when num. of dimensions
may be found by the value of $= n$ of $\frac{1}{n}$ The no. of dimensions
The imaginary roots of $x^n - 1 = 0$ being a prime no. are
the $n$ powers of $x$ from 1 to $n-1$. Also one of the imag. roots
any radical has as many values as there are units
in that index
To find the condit. $ax^2 + bx + c = 0$ may resolvable
into 2 simple factors or for $b^2$ a perfect sq.
Solve a Cubic by Cardan's method, and when that
the value enlarges only in the case in wh. 2 of the
roots are impossible.

$x^4 - \frac{x^3}{2} + \frac{3}{16} = 0$ has $= n$ $\left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right)\left(x^2 + x + \frac{3}{4}\right)$

$x^3 - 4x^2 - 6x + 12$ has a commensurable root.

$x^5 + 2x^4 - 6x^3 - 4x^2 + 13x - 6 = 0$ find how
$(x+2)(x^2-6x+6)$
often the each unity is repeated. 3 times.

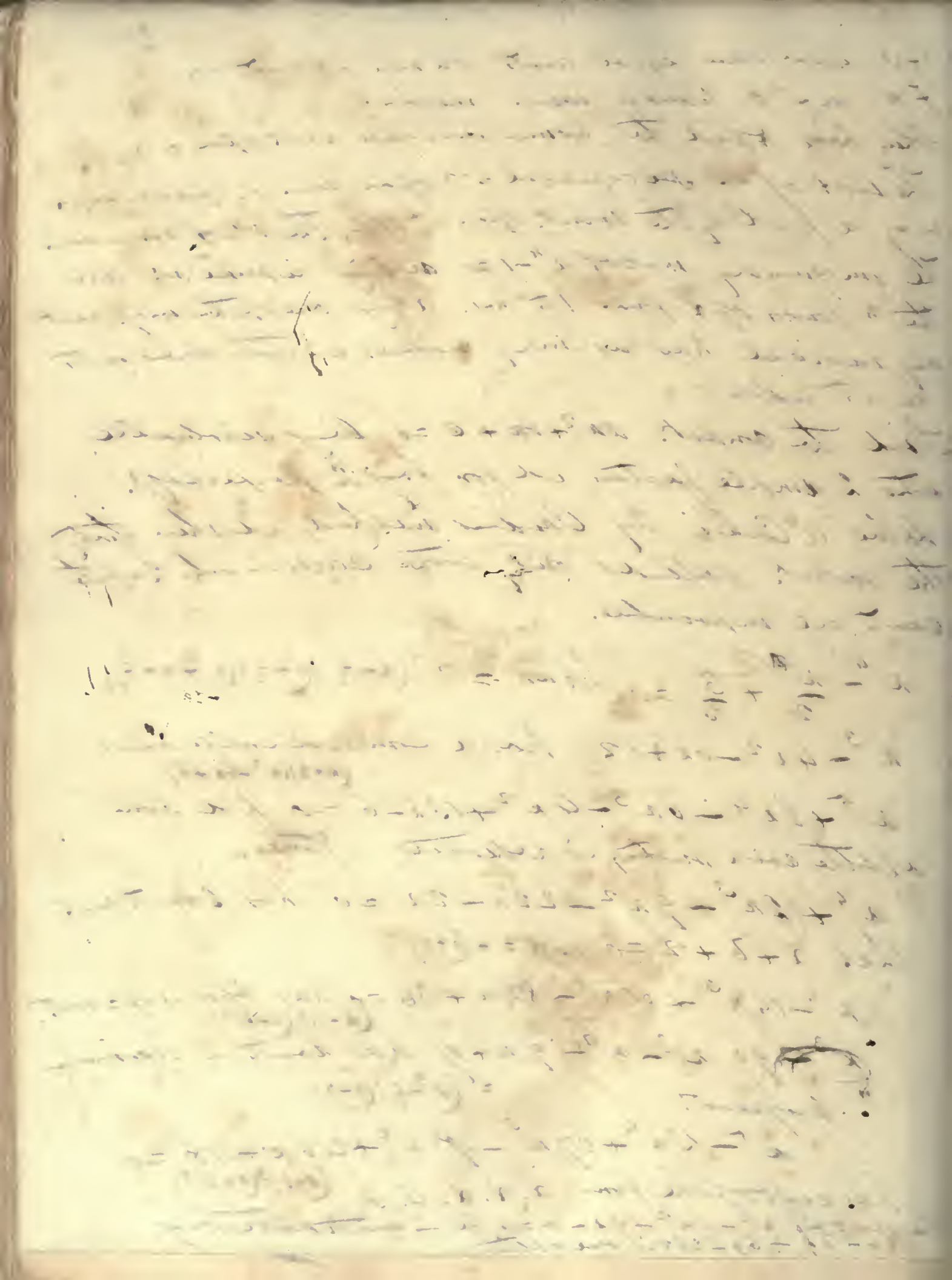
$x^4 + 2x^3 - 9x^2 - 22x - 22 = 0$ has 2 roots such
that $\alpha + \beta + 2 = 0 \therefore \alpha = -(\beta + 2)$

$x^4 - 14x^3 + 61x^2 - 84x + 36 = 0$ has 2 pairs of $=$ roots.
$(x-1)^2 \cdot (x-6)^2$.

$9x^3 - x^2 - 9x + 9$ has roots in harmonic
$= (x^2 - 9)(x - 1)$
proper.

$x^5 - 13x^4 + 67x^3 - 171x^2 + 216x - 108 = 0$
has roots of the form $\alpha, \alpha, \alpha, \beta, \beta$
$(x-3)^3 (x-2)^2$.

If the roots of $x^3 - px^2 + qx - r = 0$ be in arithmetic prog.
$9r - pq + 2p^3 = 0$ is one of them.



$$2PR\,d.y + 2P'R + y\{P\,d.R - R\,d.P\} + 2y'PR = 0$$

$$d.\left(y\sqrt{\frac{R}{P}}\right) = \sqrt{\frac{R}{P}}\,dy$$

$$= \sqrt{\frac{R}{P}}\,d.y + \frac{1}{2}y\cdot\frac{P\,d.R - R\,d.P}{P^2\sqrt{\frac{R}{P}}}$$

$$= \frac{PR\,d.y + \frac{1}{2}y\left(P\,d.R - R\,d.P\right)}{P\sqrt{PR}}$$

$$PR\,d.y + \frac{1}{2}y\left(P\,d.R - R\,d.P\right)$$

$$+ PR.\left(P + y^2R\right) = 0$$

$$\sqrt{PR}\,d.\left(y\sqrt{\frac{R}{P}}\right) + PR\{P + y^2R\} = 0$$

$$\sqrt{PR}\,d.\left(y\sqrt{\frac{R}{P}}\right) + PR\left(1 + \frac{y^2R}{P}\right) = 0$$




of the point of the particle
$OA = a$. $dx\,dy$ an element of the ellipse.
attraction of this element upon O in
direction OA is $\mu \cdot dx\,dy \cdot \frac{(a+x)}{\{(a+x)^2+y^2\}^{\frac{3}{2}}}$.

∴ whole attraction of ellipse =

$$\mu \int_x^{a\sqrt{a+c}} \int_y \frac{a+x}{\left((a+x)^2+y^2\right)^{\frac{3}{2}}}.$$

$$\int_y \frac{a+x}{\left((a+x)^2+y^2\right)^{\frac{3}{2}}} = \frac{(a+x)\,y}{\{(a+x)^2+y^2\}^{\frac{3}{2}}}$$

but $y = \frac{b}{c}\sqrt{2cx-x^2}$.

∴ attract. of $O$ =

$$\frac{\mu b}{c} \int_x^{a\sqrt{a+c}} \frac{(a+x)\sqrt{2cx-x^2}}{\sqrt{a^2+2\left(a-\frac{b^2}{c}\right)x+\left(1-\frac{b^2}{c^2}\right)x^2}}.$$

$$\frac{P.\,d.\left(y\sqrt{\frac{R}{p}}\right)}{1+\left(y\sqrt{\frac{R}{p}}\right)^2}+P\sqrt{pR}=0$$

$$d\tan^{-1}\left(y\sqrt{\frac{R}{p}}\right)+\sqrt{pR}=0$$

$$\tan^{-1}\left(y\sqrt{\frac{R}{p}}\right)+\int\sqrt{pR}=0$$

$$y=-\sqrt{\frac{p}{R}}\tan\int\sqrt{pR}$$

Ans

Let h.g.f. h'g'f' be the coords of 2 particles
the former, on an ellipsoid whose = n is

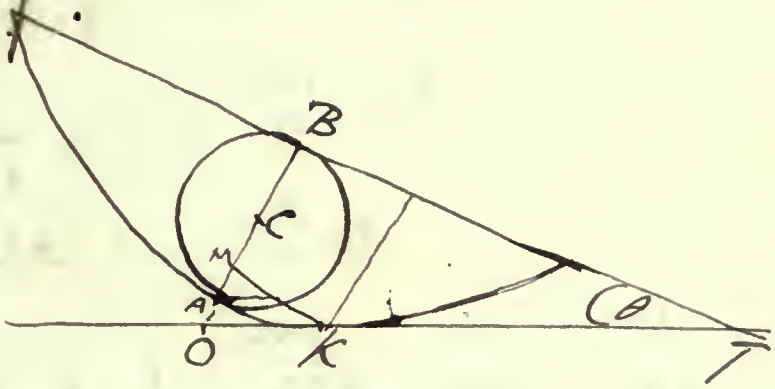
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

the latter on an ellipsoid whose
= n is $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1$

Then the attraction of the first particle
upon the second ellipsoid is the
attraction of the second particle on the
first ellipsoid // to the axes of =
as ab : αβ.

Similar expressions hold for the
attractions // to the axes of x, y.

We use of this theorem is to make
the attraction of an ellipsoid upon
an external particle depend upon
the attraction of another ellipsoid upon
an external particle.




The eqn to BT is

$$y = \cancel{\tan\theta}(x - OT)\tan\theta$$

Now $OT = Ok + kT$

Let $AK = x$, $MK = y$

Then $\theta \cot\theta = \sqrt{\frac{2a - x}{x}}$

Now $Ok = AK = 2\sqrt{2ax}$

$$kT^2 = (2a - x)^2 + \frac{(2a - x)^3}{x}$$

$$= (2a - x)^2 \cdot \frac{2a}{x}$$

$$\therefore OT = 2\sqrt{2ax} + \sqrt{2a} \cdot \frac{2a - x}{\sqrt{x}}$$

$$= \sqrt{\frac{2a}{x}} \cdot (2x + 2a - x)$$

$$= \sqrt{\frac{2a}{x}} \cdot (x + 2a)$$

$$x + y = 8$$

$$\frac{x-y}{2} + \frac{2x-y}{3} + \frac{4}{3} = 0$$

$$3x - 3y + 4x - 2y = +8$$

$$\therefore 7x - 5y = -8$$

$$7x + 7y = 56$$

$$\therefore 12y = 48$$

$$\therefore y = 4$$

and $x + y = 8$

$$\therefore x = 4$$

$$\frac{x+3}{x} + 4x = 14$$

$$\therefore x + 3 + 4x^2 = 14x$$

$$\therefore 4x^2 - 13x = -3$$

$$\therefore x^2 - \frac{13}{4}x + \frac{169}{64} = -3 + \frac{169}{64} = \frac{-192}{64} + \frac{169}{64} = -\frac{23}{64}$$

$$\therefore x - \frac{13}{8} = \pm \frac{\sqrt{23}}{8}$$

$$\therefore x = \frac{\sqrt{23}}{8} + \frac{13}{8}$$

Now $x \cot^2\theta = 2a - x$

$\therefore 2a = x \operatorname{cosec}^2\theta$

$x = 2a \sin^2\theta$

$\therefore OT = \frac{1}{\sin\theta}(2a\sin^2\theta + 2a)$

$= 2a \cdot \frac{\sin^2\theta + 1}{\sin\theta}$

$\therefore y = 2a \cdot \frac{1+\sin^2\theta}{\sin\theta} \cdot \tan\theta - x\tan\theta$

$= 2a \frac{1+\sin^2\theta}{\cos\theta} - x\tan\theta$

$a y \cos\theta + x \sin\theta = 2a(1+\sin^2\theta)$

$-y\sin\theta + x\cos^2\theta = 4a\sin\theta\cos\theta$

$\therefore x^2 - 2xx' = 4a^2\{1 + \sin^2\theta + 2\sin^2\theta\cos^2\theta\}$

$= 4a^2\{1 - \cos^4\theta + 2\sin^2\theta\}$

Gen. $S = \left\{2a + (n-1)b\right\} \frac{n}{2}$

Here $a = 4,\ b = -7,\ n = 20$

$\therefore S = \left\{8 + (20-1)-7\right\} \frac{20}{2}$

$= \left(8 + 19 \times -7\right) 10$

$= (8 - 133)\, 10$

$= -125 \times 10 = 1250$

$S = \left\{2a + (n-1)\, b\right\} \frac{n}{2}$

Here $a = \frac{n-1}{n}$, $b = -\frac{1}{n}$, $n = n$

$\therefore S = \left\{\frac{2n-2}{n} + (n-1) - \frac{1}{n}\right\} \frac{n}{2}$.

$\therefore S = \left\{\frac{2n-2}{n} + \frac{-n+1}{n}\right\} \frac{n}{2}$

$\therefore S = \frac{n-1}{n} \times \frac{n}{2}$

$\therefore S = \frac{n-1}{2}$

Ellipse theorem

$$y = 2a\{\cos\theta + \cos\theta\sin^2\theta - 2\sin^2\theta\cos\theta\}$$

$$= 2a\{2\cos\theta - 2\cos^2\theta - 2\cos\theta + 2\cos^2\theta\}$$

$$\therefore \cos^2\theta = \frac{y}{2a}$$

$$\cos^2\theta = \left\{\frac{y}{2a}\right\}^{\frac{2}{3}}$$

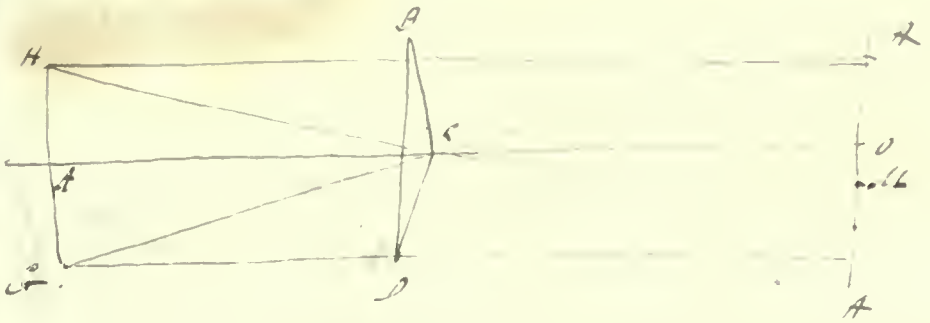
$$\therefore y\cdot\left\{\frac{y}{2a}\right\}^{\frac{1}{3}} + x\left\{1 - \left(\frac{y}{2a}\right)^{\frac{2}{3}}\right\} =$$

$$2a\left\{2 - \left(\frac{y}{2a}\right)^{\frac{2}{3}}\right\}$$

$$y\left(\frac{y}{2a}\right)^{\frac{1}{3}} + (x - 2a)\left\{1 - \left(\frac{y}{2a}\right)^{\frac{2}{3}}\right\} = 2a$$

$$y^{\frac{4}{3}} + (x - 2a)\left\{(2a)^{\frac{2}{3}} - y^{\frac{2}{3}}\right\} = (2a)^{\frac{5}{3}}$$

$$\therefore (2a)^{\frac{1}{3}}\left\{(2a)^{\frac{4}{3}} - y^{\frac{4}{3}}\right\} = (x - 2a)\left\{(2a)^{\frac{2}{3}} - y^{\frac{2}{3}}\right\}$$




Let H be the source of light from which a wave disturbance is refracted thro' the spherical surface ABC and let it be required to find the value of the displacement at the point from which, if a wave disturbance being refracted at the same surface ABC, should proceed at the same rate which proceeds from H to E, at E a point from which a wave proceeds, would refract thro' ABC would proceed at the same from H after refraction at C.

Now $\angle HCA = \angle ACD =$ [illegible]. also $\sin HCA =$ $(\mu - 1) \sin \alpha$. Let $HC = a$. $CD = b$. and in HE let a point in the wave upon which the undisturbed from which alone fall. The undisturbed from H may be represented by

$$\frac{c}{HE} \sin \frac{2\pi}{\lambda} \left( \frac{ct - HE}{\lambda} + \zeta \right).$$

and that from D may be represented by

$$\frac{c}{DE} \sin \frac{2\pi}{\lambda} \left( \frac{ct - DE}{\lambda} + \zeta \right).$$

Now since the wave from A and D must be

$$\therefore (2a)^{\frac{1}{3}}\left\{(2a)^{\frac{2}{3}} + y^{\frac{2}{3}}\right\} = x - 2a$$

$$\therefore 2a + (2a)^{\frac{1}{3}} y^{\frac{2}{3}} = x - 2a$$

$$\therefore (2a)^{\frac{1}{3}} y^{\frac{2}{3}} = x - 4a$$

in the direction $A = B$.

Now in the earth's actual surface $AA'$ and $BB'$ are very $AO = a+b$ nearly.

$\therefore$ the whole disturbance at $M$ is represented by

$$\frac{c}{a+b}\left\{\sin\frac{2\pi}{\lambda}(vt - AM + A) + \sin\frac{2\pi}{\lambda}(vt - BM + B)\right\}$$

$$= -\frac{2c}{a+b}\cos\frac{\pi}{\lambda}(AM - BM)\sin\frac{2\pi}{\lambda}\left(vt - \frac{AM + BM}{2} + \alpha\right)$$

$\therefore$ the intensity of the light is represented by

$$\frac{4c^2}{(a+b)^2}\cos^2\frac{\pi}{\lambda}(AM - BM).$$

Now $AM^2 = (a-1)^2 a^2 \sin^2\alpha + b^2$

$AM^2 = (a+b)^2 + (a-1)^2 a^2 \sin^2\alpha + 2b(a-1)a\sin^2\alpha$ with $a+b$ for $AO$.

$$\therefore AM = a+b + \frac{1}{2}\frac{(a-1)^2 a^2 \sin^2\alpha}{(a+b)}$$

$$\text{or } BM = a+b + \frac{1}{2}\cdot\frac{(a-1)^2 a^2 \sin^2\alpha}{a+b}$$

$$\therefore AM - BM = \frac{2\cdot(a-1)a\sin\alpha}{a+b}\cdot OM.$$

$\therefore$ intensity of light =

$$\frac{4c^2}{(a+b)^2}\cos^2\frac{2\pi}{\lambda}\left(\frac{a(a-1)\sin\alpha}{a+b}\right)$$

perfectly flexible chain of uniform thickness hangs in the form of a curve whose $x$ taken from the lowest point is $\frac{y}{b} = \log \sec \frac{x}{b}$. find the law of $x$ of thickness, and shew that the chain offers an equal resistance to rupture at every point.

Let $m$ = unit of mass. $T$ = tension of the chain at the lowest point. $m = \frac{T}{g} \cdot \frac{d_x^2 y}{ds}$.

Now $\frac{y}{b} = \log \sec \frac{x}{b}$ or $e^{\frac{y}{b}} = \sec \frac{x}{b}$.

$\frac{dy}{b} e^{\frac{y}{b}} = \frac{1}{b} \sec \frac{x}{b} \tan \frac{x}{b}$ or $dy = \tan \frac{x}{b}$.

$d_x^2 y = \frac{1}{b} \cdot \sec^2 \frac{x}{b}$.

$ds = \sqrt{1 + (d_x y)^2} = \sqrt{1 + \tan^2 \frac{x}{b}} = \sec \frac{x}{b}$.

$\therefore m = \frac{T}{bg} \cdot \sec \frac{x}{b} = \frac{T}{bg} \cdot e^{\frac{y}{b}}$.




at $I = T \sec \theta$

$= T \sec \frac{x}{b}$

$\frac{\text{tens at } P}{x} = \text{Constant}$

$$d_s y = \frac{d_x y}{\sqrt{1+(d_x y)^2}}$$

$$d_s x = \frac{1}{\sqrt{1+(d_x y)^2}}$$

$$\frac{x d_x y + y}{\sqrt{1+(d_x y)^2}} = \sqrt{x^2+y^2}$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

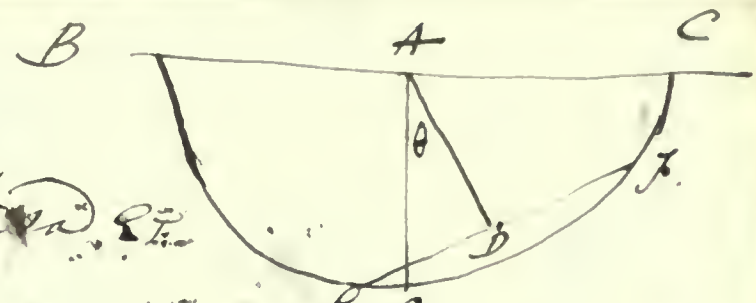
$$\frac{x d_\theta y + y d_\theta x}{\sqrt{(d_\theta x)^2+(d_\theta y)^2}} = r$$

$$\left.\begin{aligned} d_\theta y &= r\cos\theta + \sin\theta\, d_\theta r \\ d_\theta x &= -r\sin\theta + \cos\theta\, d_\theta r \end{aligned}\right\}$$

$$x d_\theta y + y d_\theta x = r^2\cos^2\theta + d_\theta r \sin 2\theta \cdot r$$

$$(d_\theta x)^2 + (d_\theta y)^2 = r^2 + (d_\theta r)^2$$

$$r\cos^2\theta + d_\theta r \cdot \sin 2\theta = \sqrt{r^2+(d_\theta r)^2}$$




B & C a vertical ∥ⁿ
of the bowl: passing thro' the C.G.
of D being the C.G. of the rod which is the middle
point AD = c. ∠ CAD = θ. (CA a vertical line).

∴ for the motion of the rod we have

$$m' k^2 \frac{d^2\theta}{dt^2} = -g m c \sin\theta \qquad k^2 = \text{rad of gyn. about } A$$

$$k^2 \frac{d^2\theta}{dt^2} + g c \sin\theta = 0$$

If θ is small we may take $\sin\theta = \theta$.

$$\therefore k^2 \frac{d^2\theta}{dt^2} + g c \theta = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{g c}{k^2}\theta = 0 \qquad \frac{d^2\theta}{dt^2} + n^2\theta = 0$$

$$\theta = A \sin\left\{\sqrt{\frac{g c}{k^2}}\, t + B\right\} \qquad \frac{\pi}{n} \qquad \frac{\pi k}{\sqrt{g c}}$$

If 2a = length of the rod. then rad of gyn. about an axis thro' D ⊥ to the rod $= \frac{a^2}{3}$.

$$\therefore k^2 = c^2 + \frac{a^2}{3} = \frac{3c^2 + a^2}{3}$$

$$\therefore \theta = A \sin\left\{\sqrt{\frac{3 g c}{3c^2 + a^2}}\, t + B\right\}$$

$$r^2\cos^2 2\theta + 2r\,d_\theta r \sin 2\theta\cos 2\theta + \sin^2 2\theta\,(d_\theta r)^2 = r^2 + (d_\theta r)^2$$

$$r^2\sin^2 2\theta + (d_\theta r)^2\cos^2 2\theta = 2r\,d_\theta r \sin 2\theta\cos 2\theta$$

$$\left(r\sin 2\theta - d_\theta r\cos 2\theta\right)^2 = 0$$

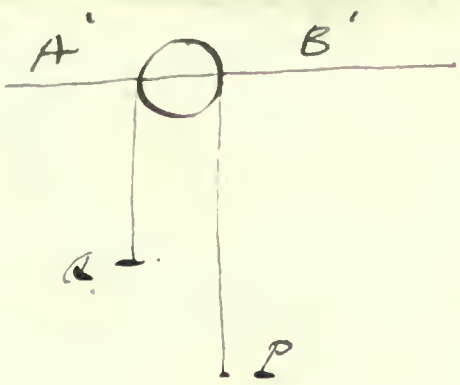
$$r\sin 2\theta = d_\theta r\cos 2\theta$$

$$\frac{d_\theta r}{r} = \frac{1}{2}\,\frac{d_\theta \cos 2\theta}{\cos 2\theta}$$

$$\log_e r = \log_e C - \frac{1}{2}\log_e \cos 2\theta$$

$$r = \frac{C}{\sqrt{\cos 2\theta}}$$

$$r^2\cos 2\theta = C^2$$




let $m, m'$ be the masses of P & Q.

~~∴ force on P = g(m − m')~~

∴ force on P = $mg - T$.

acc. force on P = $g - \frac{T}{m}$.

acc. force on Q = $\frac{T}{m'} - g$.

∴ $g - \frac{T}{m} = \frac{T}{m'} - g$.

$2g = T\left(\frac{1}{m'} + \frac{1}{m}\right) = \frac{m+m'}{mm'} T$.

∴ $T = \frac{2g\,mm'}{m+m'}$.

which gives T.

also acc. force on P = $g - \frac{T}{m} = g - \frac{2gm'}{m+m'}$

$= \frac{(m-m')g}{m+m'}$

= acc. force on Q.

~~acc. force on Q = $\frac{T}{m'} - g$.~~

$= g\left(\frac{2m}{m+m'} - 1\right) = g\left(\frac{m-m'}{m+m'}\right)$

Let $v$ be the velocities of P and Q at the time the string was cut.

$x$ = dist. from plane A'B' at any time

$a$ = dist. at the time the string was cut.

for the motion of P. $\frac{d^2x}{t} = +g$

$\therefore \frac{dx}{t} = +gt + v$

$x = a + vt + \frac{1}{2}gt^2$.

Similarly for the motion of Q. $\frac{dx'}{t} = gt - v$ $\therefore x' = b - vt + \frac{1}{2}gt^2$.

$$x^3 d_x y = (y - x d_x y)^2$$

$$d_x \left(\frac{y}{x}\right) = \frac{y - x d_x y}{x^2}$$

$$\therefore x^3 d_x y = x^4 \left(d_x \frac{y}{x}\right)^2$$

$$x d_x y = x^2 \left(d_x \frac{y}{x}\right)^2$$

$$d_x \left(x^2 d_x \frac{y}{x}\right) = x^2 \left(d_x \frac{y}{x}\right)^2$$

$$x^2 d_x \left(\frac{y}{x}\right) = z$$

$$d_x z = x^2 \frac{z^2}{x^4} = \frac{z^2}{x^2}$$

$$\frac{1}{z} = \frac{1}{x} + C$$

$$x = z + C x z$$

$$= x^2 d_x \left(\frac{y}{x}\right)(1 + C x)$$

$$d_x \left(\frac{y}{x}\right) = \frac{1}{x(1 + C x)} = \frac{\frac{1}{x^2}}{\frac{1}{x} + C}$$

$$\frac{y}{x} = -\log_e \left(\frac{1}{x} + C\right) = \log_e \frac{x}{C x + 1}$$

4) $x^9+x^8-9x^7+3x^6-8x^5-8x^4+3x^3-9x^2+x+1=$

divide by $x+1$

$1+1-9+3-8-8+3-9+1+1$

$-1 \quad 2 \quad -4 \quad 4 \quad -12 \quad 20$

$1 \quad 0 \quad -9 \quad 12 \quad -20 \quad 12 \quad -9 \quad 0 \quad 1 \quad 0$

and the reduced equation is

$x^8-9x^6+12x^5-20x^4+12x^3-9x^2+1=0$

$x^4-9x^2+12x-20+\frac{12}{x}-\frac{9}{x^2}+\frac{1}{x^4}=$

or $\left(x^4+\frac{1}{x^4}\right)-9\left(x^2+\frac{1}{x^2}\right)+12\left(x+\frac{1}{x}\right)-20=$

put $x+\frac{1}{x}=z$

$\therefore x^2+\frac{1}{x^2}=z^2-2$

$x^3+\frac{1}{x^3}=z^3-2z-z$

$=z^3-3z$

$x^4+\frac{1}{x^4}=z^4-3z^2-z^3+3z$

$=z^4-z^3-3z^2+3z$

$\therefore z^4-z^3-3z^2+3z-9(z^2-2)+12z-20=$

$z^4-z^3-12z^2+15z-2=0$

$$\tan(\theta - \varphi) = \frac{\frac{r}{\rho^2}\frac{d\theta}{d\rho}}{1 + \frac{1}{\rho^2}\frac{dr}{d\rho}}$$

If $\tan\varphi = k$




$$f(r, \rho) = 0$$

$$r^2 = \rho^2 + a^2 - 2\rho a \cos\theta$$

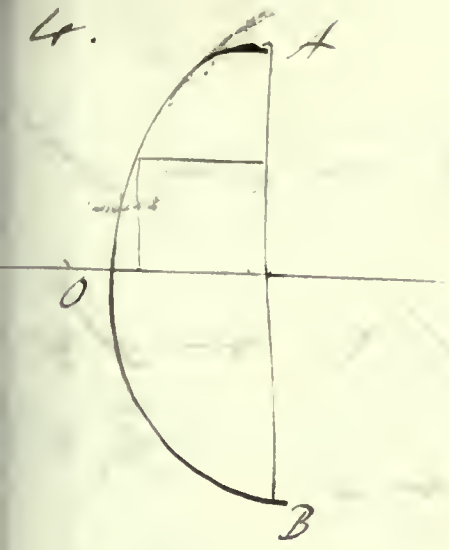
$$r - \{\rho, \sqrt{\rho^2 + a^2 - 2\rho a\cos\theta}\} = 0$$

$$r + n\rho = c$$

$$n\rho + \sqrt{\rho^2 + a^2 - 2\rho a\cos\theta} = c$$

$$\rho^2 + a^2 - 2\rho a\cos\theta = c^2 - 2cn\rho + n^2\rho^2$$

4.




$\int_s$ the rad. of gyration about AB

$$\Sigma m(2a-x)^2 = \int_s (2a-x)^2$$

$$= \int_x (2a-x)^2 d_x s$$

$s' = 8ax$ $\quad a =$ rad. of gyration ⊙

$$s\, d_x s = 4a$$

$$d_x s = \frac{4a}{2\sqrt{2ax}} = \sqrt{\frac{2a}{x}}$$

mom. of int. $= \int_x \sqrt{2a}\,\frac{(2a-x)^2}{\sqrt{x}}$

$$= \sqrt{2a}\int_x \left(\frac{4a^2}{\sqrt{x}} - 4a\sqrt{x}\cdot x + x^{\frac{3}{2}}\right)$$

$$= \sqrt{2a}\left\{8a^2 x^{\frac{1}{2}} - \frac{8a}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}}\right\} + C$$

$$= \sqrt{2a}\left\{8a^2\sqrt{2a} - \frac{16a^2}{3}\sqrt{2a} + \frac{8a^2}{5}\sqrt{2a}\right\}$$

$$= 2a\left(8a^2 - \frac{16a^2}{3} + \frac{8a^2}{5}\right)$$

$$= 2a^3\left(\frac{120 - 80 + 24}{15}\right)$$

$$= \frac{320}{15}a^3$$

$\therefore (h^2 - 1)\rho^2 - 2(hc + ab\cos\theta)\rho + c^2 - a^2 = 0$

find $\rho$ in terms of $\theta$ & $\rho$

& $\tan SPy = \frac{}{}$

$h = .1$

$\rho = \frac{c^2 - a^2}{2(hc + ab\cos\theta)}$

= to Conic Section.

$$mk = \Sigma \delta s . r = \int \frac{a\,dr}{r\sqrt{2ar - r^2}}$$

$$= -a\sqrt{2ar - r^2} + a^2 m^{-1}\frac{r}{a}$$

$$\therefore l = \frac{mk'}{mk} = 2a$$ which is independent

of the length of the arc.

$$x = \frac{\pi}{2} + \theta$$

$$\int \frac{1}{-\sin\theta \cdot \sqrt{1+\cos\theta}} = -\int \frac{1}{\sin\theta \cdot \sqrt{2} \cdot \cos\frac{\theta}{2}}$$

$$= -\frac{1}{\sqrt{2}} \int \frac{1}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= -\frac{1}{2\sqrt{2}} \int \frac{\sec^2\frac{\theta}{2}}{\tan\frac{\theta}{2}}$$

$$= -\frac{1}{2\sqrt{2}} \int \left\{ \frac{1}{\tan\frac{\theta}{2}} + \frac{\sec^2\frac{\theta}{2}}{\tan\frac{\theta}{2}} \right\}$$

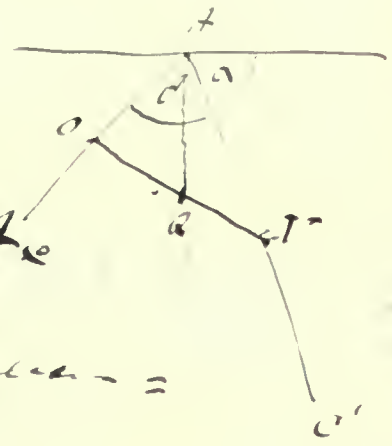
$$= -\frac{1}{\sqrt{2}} \int_{\frac{\theta}{2}} \left\{ \frac{1}{\tan\frac{\theta}{2}} - \frac{2\,d_\theta \cos\frac{\theta}{2}}{\cos\frac{\theta}{2}} \right\}$$

$$= -\frac{1}{\sqrt{2}} \left\{ \log_e \tan\frac{\theta}{4} + \frac{2}{\cos\frac{\theta}{2}} \right\}$$

$$= -\frac{1}{\sqrt{2}} \left\{ \log_e \tan\left(\frac{x}{4} - \frac{\pi}{8}\right) + 2\sec\left(\frac{x}{2} - \frac{\pi}{4}\right) \right\}$$

$AB = a$. $AC' = b$ the arms.

$k$ = dist. of centre of grav. of $AB$, $AC'$ from $A$. $h_1$ = vertical height of centre of gravity above horizontal axis through $A$




$\therefore$ Velocity of projection for greatest range $= \frac{h^2}{h_1}$

Where $h^2 = \frac{1}{3}(a^3 + b^3)$.

Also of O.V. being middle point of $AC'$, and be joined. And Q be such that

$OQ = \frac{a}{a+b} \cdot OV$. Q is the centre of gravity of the rod. and $AQ$ is vertical in position of greatest range.

$$OV = \sqrt{\frac{a^2}{4} + \frac{b^2}{4} - \frac{ab}{2}\cos\theta}.$$

$$OQ = \frac{a}{a+b} \cdot \sqrt{\frac{a^2}{4} + \frac{b^2}{4} - \frac{ab}{2}\cos\theta}.$$

$$AQ^2 = \frac{b^2}{4} + OQ^2 + \frac{b^2}{2} \cdot OQ \cdot \cos AOV.$$

$$= \frac{b^2}{4} + OQ^2 + \frac{b^2}{2} \cdot \frac{a}{a+b}\sqrt{\frac{a^2}{4} + \frac{b^2}{4} - \frac{ab}{2}\cos\theta} \cdot \left\{ \frac{\frac{b^2}{4} + OV^2 - \frac{a^2}{4}}{2 \cdot OV} \right\}.$$

$$= \frac{b^2}{4} + OQ^2 + \frac{b^2}{2} \cdot \frac{a}{a+b} \left\{ \frac{b^2}{4} + OV^2 - \frac{a^2}{4} \right\}.$$

which by reduction becomes $\frac{1}{2} \cdot \sqrt{a^4 + 2a^2b^2\cos\theta + b^4}$.

$$\therefore \frac{h^2}{h_1} = \frac{2}{3} \cdot \frac{a^3 + b^3}{\sqrt{a^4 + 2a^2b^2\cos\theta + b^4}}$$

$$d_2 y = \sin(mx + ny)$$

$$\text{Let } mx + ny = z$$

$$\text{Then } m + n d_1 y = d_1 z$$

$$\therefore \frac{d_1 z - m}{n} = \sin z$$

$$d_1 z = m + n \sin z$$

$$x = \int_z \frac{1}{m + n \sin z}$$

$$= \frac{1}{n} \int_z \frac{1}{\frac{m}{n} + \sin z}$$

$$= \frac{1}{n} \int_z \frac{\sec^2 \frac{z}{2}}{\frac{m}{n}\left(\tan^2 \frac{z}{2} + 1\right) + 2 \cdot \tan \frac{z}{2}}$$

$$= \frac{1}{m} \int_z \frac{\sec^2 \frac{z}{2}}{\tan^2 \frac{z}{2} + \frac{2n}{m} \tan \frac{z}{2} + 1}$$

$$= \frac{2}{m} \int \frac{\frac{1}{2} \tan\frac{z}{2}}{\left(\tan\frac{z}{2} + \frac{n}{m}\right)^2 + 1 - \frac{n^2}{m^2}}$$

$$\therefore = \frac{2}{m} \cdot \frac{m}{\sqrt{m^2 - n^2}} \tan^{-1} \frac{n \tan\frac{z}{2} + n}{\sqrt{m^2 - n^2}}$$

$$= \frac{2}{\sqrt{m^2 - n^2}} \tan^{-1} \frac{n \tan\left(\frac{x+y}{2}\right) + n}{\sqrt{m^2 - n^2}}$$

The = n Salte H.c paraboloid is

$$x = \frac{y^2}{l} - \frac{z^2}{l'}$$

for a plane ⊥ Salte axis x is some constant = k suppose.

∴ the = n Salte ∫ n is $\frac{z^2}{kl'} - \frac{y^2}{kl} = 1$

which represents an H. with major axis = $\sqrt{kl'}$, minor axis = $\sqrt{kl}$. the minor axis being ∥ Salte axis ofy.

From the form of the = n. which contains only even powers of z and y the origin is the centre of the curve. ~~The axes are the axes of z. & y of the~~

~~parabola~~

The trace on yz is. $y = \pm \sqrt{\frac{l}{l'}}\, z$.

which represents 2 straight lines.

$$P\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{y}{x}$$

$$P\frac{d^2y}{dx^2} - \frac{d}{dx}\left(\frac{y}{x}\right) + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x} = 0$$

Let $x\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{1}{x}\frac{dy}{dx} - \frac{y}{x} = z$

$$P\frac{dz}{dx} + z = 0$$

$$\frac{dz}{dx} + \frac{1}{P}z = 0$$

Integrating factor $= \varepsilon^{+\int\frac{1}{P}dx}$

$$\frac{d}{dx}\left(z\varepsilon^{+\int\frac{1}{P}dx}\right) = 0$$

$$z = C\varepsilon^{-\int\frac{1}{P}dx} = x\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\therefore y = Cx\int\frac{1}{x^2}\varepsilon^{-\int\frac{1}{P}dx}$$

which is the solution of the Equation

4.

Let $\theta$ $\theta'$ be the $\angle$s which the normal to a front makes with the optic axes.

Then the normal makes $\angle$s whose $\cos$s $l, m, n$ with the axes & the optic axes make $\angle$s whose $\cos$s

$\pm \sqrt{\frac{a^2-b^2}{a^2-c^2}}$ 0 $\pm \sqrt{\frac{b^2-c^2}{a^2-c^2}}$ with the axes

$$\therefore \cos\theta = l\sqrt{\frac{a^2-b^2}{a^2-c^2}} + n\sqrt{\frac{b^2-c^2}{a^2-c^2}}$$

$$\text{or } (a^2-c^2)\cos\theta = l\sqrt{(a^2-b^2)(a^2-c^2)} + n\sqrt{(b^2-c^2)(a^2-c^2)}$$

Now $$\frac{l^2}{v^2-a^2} + \frac{m^2}{v^2-b^2} + \frac{n^2}{v^2-c^2} = 0$$

$$\text{or } l^2(v^2-b^2)(v^2-c^2) + m^2(v^2-a^2)(v^2-c^2) + n^2(v^2-a^2)(v^2-b^2) = 0$$

$$\text{or } l^2(v^2-a^2+a^2-b^2)(v^2-a^2+a^2-c^2) + m^2(v^2-a^2)(v^2-a^2+a^2-c^2) + n^2(v^2-a^2)(v^2-a^2+a^2-b^2) = 0$$

$$\text{or } (v^2-a^2)^2 + A(v^2-a^2) + l^2(a^2-b^2)(a^2-c^2) = 0$$

$\therefore$ If $v_1^2$ $v_2^2$ be the 2 values of $v^2$

$$\frac{dp}{dx} - \frac{1}{x} p = x$$

$$\frac{\frac{dp}{dx}}{x} - \frac{1}{x^2} p = 1$$

$$\frac{d}{dx}\left(\frac{p}{x}\right) = 1$$

$$\frac{p}{x} = x + C$$

$$\frac{dy}{dx} = x^2 + Cx$$

$$y = \frac{x^3}{3} + \frac{Cx^2}{2} + C_1$$

4/2

$$(v_1^2 - a^2)(v_2^2 - a^2) = b^2(a^2 - b^2)(a^2 - c^2)$$

Simily $(v_1^2 - c^2)(v_2^2 - c^2) = a^2(b^2 - c^2)(a^2 - c^2)$

$$\therefore (a^2 - c^2)\cos\theta = \sqrt{(v_1^2 - a^2)(v_2^2 - a^2)} + \sqrt{(v_1^2 - c^2)(v_2^2 - c^2)}$$

Simily $(a^2 - c^2)\cos\theta' = \sqrt{(v'^2 - a^2)(v_2^2 - a^2)} - \sqrt{(v_1^2 - c^2)(v_2^2 - c^2)}$

Now $\sin^2\theta \cdot \sin^2\theta' = 1 - (\cos^2\theta + \cos^2\theta') + \cos^2\theta\cos^2\theta'$

$$\therefore (a^2 - c^2)^2 \sin^2\theta \sin^2\theta' = (a^2 - c^2)^2 - (a^2 - c^2)^2(\cos^2\theta + \cos^2\theta') + (a^2 - c^2)^2 \cos^2\theta \cos^2\theta'$$

$$(a^2 - c^2)^2(\cos^2\theta + \cos^2\theta') = 2(v_1^2 - a^2)(v_2^2 - a^2) + 2(v_1^2 - c^2)(v_2^2 - c^2)$$

$$= 4v_1^2 v_2^2 - 2a^2(v_1^2 + v_2^2) - 2c^2(v_1^2 + v_2^2) + 2(a$$

$$= 4v_1^2 v_2^2 - 2(a^2 + c^2)(v_1^2 + v_2^2) + 2(a^4 + c^4)$$

$$dy = x(y^2+1)$$

$$\frac{dy}{y^2+1} = x$$

$$\tan^{-1} y = \frac{x^2}{2} + C$$

$$x^2 + y^2 + (a+x)y\,dy = 0$$

$$x^2 + y^2 + \frac{1}{2}(a+x)\,dy^2 = 0$$

$$d(y^2) + \frac{2}{a+x}\,y^2 = -\frac{2x^2}{a+x}$$

$$\text{Integrating factor} = e^{\int \frac{2}{a+x}}$$

$$= (a+x)^2$$

$$(a+x)^2 d(y^2) + 2y^2(a+x) = -2x^2(a+x)$$

$$d\{(a+x)^2 y^2\} = -2x^2(a+x)$$

$$(a+x)^2 y^2 = C - \frac{2ax^3}{3} - \frac{x^4}{2}$$

3/

Let $Y_m . Z_n$ be any 2 Laplaces coeffts of the $m$th and $n$th orders respectively.

Since $Z_n$ is a Laplaces coefft of the $n$th order it satisfies the $= n$.

$$n(n+1) \int_0^{2\pi} \int_{-1}^{+1} Y_m . Z_n . d\mu' d\varphi' =$$

$$- \int_0^{2\pi} \int_{-1}^{+1} Y_m \left\{ d\mu . (1-\mu^2) d_\mu^2 Z_n + \frac{1}{1-\mu^2} . d_\varphi^2 . Z_n \right\} d\mu' d\varphi'$$

Now integrate by parts observing that $1-\mu^2 = 0$ for both limits we have.

$$\int_{\mu'}^{+1} Y_m . d\mu \left( 1-\mu^2 \right) d_\mu Z_n \Big] = - \int_{\mu'}^{+1} d_\mu Y_m (1-\mu^2) . d_\mu Z_n$$

Similarly for we have.

$$\int_{\varphi'}^{2\pi} Y_m \frac{1}{1-\mu^2} . d^2 Z_n = - \int_{\varphi'}^{2\pi} d_\varphi Y_m \frac{1}{(1-\mu^2)} d_\varphi Z_n$$

Similar expressions occur for the other quantity $Y_m$. By subtraction we have.

$$\therefore \left\{ n(n+1) - m(m+1) \right\} \int_0^{2\pi} \int_{-1}^{+1} Y_m Z_n \Big\} = 0.$$

and if $n$ does not $= m$. this can only be satisfied by $\int_0^{2\pi} \int_{-1}^{+1} Y_m Z_n = 0$.




$f(p, \rho) = 0$

$p = SY \quad \rho = SZ$

Now $SP \cdot SZ = SY^2$

$$\rho \cdot p_1 = b^2 = \rho_1^2$$

$$\left.\begin{aligned} \therefore \quad p &= \rho_1 \\ \therefore \quad \rho &= \frac{\rho_1^2}{p_1} \end{aligned}\right\}$$

$$\therefore f\left\{\rho_1, \frac{\rho_1^2}{p_1}\right\} = 0$$

$\rho_1 = \phi(p, \rho)$ polar = $n^{th}$ pedal curve. $\frac{1}{\rho_1} = \phi\left(p, \frac{1}{\rho}\right)$ inverse = $n^{th}$ inverse locus of $Y$.

4/2

$$mk = \int_S (2a - x) = \int_h (2a - x)\, d_2 S$$

$$= \sqrt{2a}\left(4a\sqrt{h} - \frac{2}{3}h^{\frac{3}{2}}\right) + C$$

$$= \rho a^2 - \frac{\rho a^2}{3} = \frac{16a^2}{3}$$

$$l = \frac{mk_1^2}{mk} = \frac{32\rho \cdot a^2}{15 \times \frac{16}{3}} = \frac{32\rho}{\rho_0} \cdot a$$

$$= 2{\cdot}1\, a$$

now $\alpha$ = length of Cycloid

$$= 4\sqrt{2ah} = \rho a \qquad a = \frac{\alpha}{\rho}$$

$$l = \frac{2{\cdot}1}{\rho} \cdot \alpha$$

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